Recall from Section 1.3 that statements that can be proved are called theorems. To understand the formal proof of a theorem, we begin by considering the terms hypothesis and conclusion. The hypothesis of a statement describes the given situation (Given), whereas the conclusion describes what you need to establish (Prove). When a statement has the form “If H, then C,” the hypothesis is H and the conclusion is C. Some theorems must be reworded to fit into “If . . . , then . . .” form so that the hypothesis and conclusion are easy to recognize.

**EXAMPLE 1**

Give the hypothesis H and conclusion C for each of these statements.

a) If two lines intersect, then the vertical angles formed are congruent.

b) All right angles are congruent.

c) Parallel lines do not intersect.

d) Lines are perpendicular when they meet to form congruent adjacent angles.

**SOLUTION**

a) As is

H: Two lines intersect.
C: The vertical angles formed are congruent.

b) Reworded

If two angles are right angles, then these angles are congruent.

H: Two angles are right angles.
C: The angles are congruent.

c) Reworded

If two lines are parallel, then these lines do not intersect.

H: Two lines are parallel.
C: The lines do not intersect.

d) Reordered

When (if) two lines meet to form congruent adjacent angles, these lines are perpendicular.

H: Two lines meet to form congruent adjacent angles.
C: The lines are perpendicular.

Why do we need to distinguish between the hypothesis and the conclusion? For a theorem, the hypothesis determines the Given and the Drawing. The Given provides a description of the Drawing’s known characteristics. The conclusion (Prove) determines the relationship that you wish to establish in the Drawing.

**THE WRITTEN PARTS OF A FORMAL PROOF**

The five necessary parts of a formal proof are listed in the following box in the order in which they should be developed.
The most difficult aspect of a formal proof is the thinking process that must take place between parts 4 and 5. This game plan or analysis involves deducing and ordering conclusions based on the given situation. One must be somewhat like a lawyer, selecting the claims that help prove the case while discarding those that are superfluous. In the process of ordering the statements, it may be beneficial to think in reverse order, like so:

The Prove statement would be true if what else were true?

The final proof must be arranged in an order that allows one to reason from an earlier statement to a later claim by using deduction (perhaps several times). Where principle \( P \) has the form “If \( H \), then \( C \),” the logical order follows.

\[
\begin{array}{c}
H: \text{hypothesis} \\
P: \text{principle} \\
\therefore C: \text{conclusion}
\end{array}
\]

Consider the following theorem, which was proved in Example 1 of Section 1.6.

**THEOREM 1.6.1**

If two lines are perpendicular, then they meet to form right angles.

**EXAMPLE 2**

Write the parts of the formal proof of Theorem 1.6.1.

**SOLUTION**

1. State the theorem.
   
   *If two lines are perpendicular, then they meet to form right angles.*

2. The hypothesis is \( H \): Two lines are perpendicular. Make a Drawing to fit this description. (See Figure 1.65.)

3. Write the Given statement, using the Drawing and based on the hypothesis \( H \):
   
   Two lines are \( \perp \).
   
   *Given:* \( \overline{AB} \perp \overline{CD} \) intersecting at \( E \)

4. Write the Prove statement, using the Drawing and based on the conclusion \( C \):
   
   They meet to form right angles.
   
   *Prove:* \( \angle AEC \) is a right angle.

5. Construct the Proof. This formal proof is found in Example 1, Section 1.6.
CONVERSE OF A STATEMENT

The converse of the statement “If $P$, then $Q$” is “If $Q$, then $P$.” That is, the converse of a given statement interchanges its hypothesis and conclusion. Consider the following:

Statement: If a person lives in London, then that person lives in England.

Converse: If a person lives in England, then that person lives in London.

As shown above, the given statement is true, whereas its converse is false. Sometimes the converse of a true statement is also true. In fact, Example 3 presents the formal proof of Theorem 1.7.1, which is the converse of Theorem 1.6.1.

Once a theorem has been proved, it may be cited thereafter as a reason in future proofs. Thus, any theorem found in this section can be used for justification in proof problems found in later sections.

The proof that follows is nearly complete! It is difficult to provide a complete formal proof that explains the “how to” and simultaneously presents the final polished form. Example 3 illustrates the polished proof. You do not see the thought process and the scratch paper needed to piece this puzzle together.

The proof of a theorem is not unique! For instance, students’ Drawings need not match, even though the same relationships should be indicated. Certainly, different letters are likely to be chosen for the Drawing that illustrates the hypothesis.

**Example 3**

Give a formal proof for Theorem 1.7.1.

If two lines meet to form a right angle, then these lines are perpendicular.

**GIVEN:** $AB$ and $CD$ intersect at $E$ so that $\angle AEC$ is a right angle (Figure 1.66)

**PROVE:** $AB \perp CD$

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $AB$ and $CD$ intersect so that $\angle AEC$ is a right angle</td>
<td>Given</td>
</tr>
<tr>
<td>2. $m \angle AEC = 90$</td>
<td>If an $\angle$ is a right $\angle$, its measure is 90</td>
</tr>
<tr>
<td>3. $\angle AEB$ is a straight $\angle$, so $m \angle AEB = 180$</td>
<td>If an $\angle$ is a straight $\angle$, its measure is 180</td>
</tr>
<tr>
<td>4. $m \angle AEC + m \angle CEB = m \angle AEB$</td>
<td>Angle-Addition Postulate</td>
</tr>
<tr>
<td>(2), (3), (4) 5. $90 + m \angle CEB = 180$</td>
<td>Substitution</td>
</tr>
<tr>
<td>(5) 6. $m \angle CEB = 90$</td>
<td>Subtraction Property of Equality</td>
</tr>
<tr>
<td>(2), (6) 7. $m \angle AEC = m \angle CEB$</td>
<td>Substitution</td>
</tr>
<tr>
<td>8. $\angle AEC \equiv \angle CEB$</td>
<td>If two $\angle$s have $\equiv$ measures, the $\angle$s are $\equiv$</td>
</tr>
<tr>
<td>9. $AB \perp CD$</td>
<td>If two lines form $\equiv$ adjacent $\angle$s, these lines are $\perp$</td>
</tr>
</tbody>
</table>

Because perpendicular lines lead to right angles, and conversely, a square (see Figure 1.66) may be used to indicate perpendicular lines or a right angle.
Several additional theorems are now stated, the proofs of which are left as exercises. This list contains theorems that are quite useful when cited as reasons in later proofs. A formal proof is provided only for Theorem 1.7.6.

**Theorem 1.7.2**
If two angles are complementary to the same angle (or to congruent angles), then these angles are congruent.

See Exercise 27 for a drawing describing Theorem 1.7.2.

**Theorem 1.7.3**
If two angles are supplementary to the same angle (or to congruent angles), then these angles are congruent.

See Exercise 28 for a drawing describing Theorem 1.7.3.

**Theorem 1.7.4**
Any two right angles are congruent.

**Theorem 1.7.5**
If the exterior sides of two adjacent acute angles form perpendicular rays, then these angles are complementary.

For Theorem 1.7.5, we create an informal proof called a picture proof. Although such a proof is less detailed, the impact of the explanation is the same! This is the first of several “picture proofs” found in this textbook. In Figure 1.67, the square is used to indicate that $\overline{BA} \, \perp \, \overline{BC}$.

**Picture Proof of Theorem 1.7.5**

| Given: $\overline{BA} \, \perp \, \overline{BC}$ |
| Prove: $\angle 1$ and $\angle 2$ are complementary |
| Proof: With $\overline{BA} \, \perp \, \overline{BC}$, we see that $\angle 1$ and $\angle 2$ are parts of a right angle. |
| Then $m\angle 1 + m\angle 2 = 90^\circ$, so $\angle 1$ and $\angle 2$ are complementary. |

*Figure 1.67*

**Strategy for Proof**

*General Rule:* The last reason explains why the last statement must be true. Never write the word “Prove” for any reason in a proof.

*Illustration:* The final reason in the proof of Theorem 1.7.6 is the definition of supplementary angles: If the sum of measures of two angles is $180^\circ$, the angles are supplementary.
EXAMPLE 4

Study the formal proof of Theorem 1.7.6.

Theorem 1.7.6

If the exterior sides of two adjacent angles form a straight line, then these angles are supplementary.

Given: \( \angle 3 \) and \( \angle 4 \) and \( \overline{EG} \) (Figure 1.68)

Prove: \( \angle 3 \) and \( \angle 4 \) are supplementary

Proof

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 3 ) and ( \angle 4 ) and ( \overline{EG} )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( m\angle 3 + m\angle 4 = m\angle EFG )</td>
<td>2. Angle-Addition Postulate</td>
</tr>
<tr>
<td>3. ( \angle EFG ) is a straight angle</td>
<td>3. If the sides of an ( \angle ) are opposite rays, it is a straight ( \angle )</td>
</tr>
<tr>
<td>4. ( m\angle EFG = 180 )</td>
<td>4. The measure of a straight ( \angle ) is 180</td>
</tr>
<tr>
<td>5. ( m\angle 3 + m\angle 4 = 180 )</td>
<td>5. Substitution</td>
</tr>
<tr>
<td>6. ( \angle 3 ) and ( \angle 4 ) are supplementary</td>
<td>6. If the sum of the measures of two ( \angle )s is 180, the ( \angle )s are supplementary</td>
</tr>
</tbody>
</table>

The final two theorems in this section are stated for convenience. We suggest that the student make drawings to illustrate Theorem 1.7.7 and Theorem 1.7.8.

Theorems 1.7.7 and 1.7.8

If two line segments are congruent, then their midpoints separate these segments into four congruent segments.

If two angles are congruent, then their bisectors separate these angles into four congruent angles.

Exercises 1.7

In Exercises 1 to 6, state the hypothesis \( H \) and the conclusion \( C \) for each statement.

1. If a line segment is bisected, then each of the equal segments has half the length of the original segment.
2. If two sides of a triangle are congruent, then the triangle is isosceles.
3. All squares are quadrilaterals.
4. Every regular polygon has congruent interior angles.
5. Two angles are congruent if each is a right angle.
6. The lengths of corresponding sides of similar polygons are proportional.
7. Name, in order, the five parts of the formal proof of a theorem.
8. Which part (hypothesis or conclusion) of a theorem determines the a) Drawing? b) Given? c) Prove?
9. Which part (Given or Prove) of the proof depends upon the 
   a) hypothesis of theorem? 
   b) conclusion of theorem? 
10. Which of the following can be cited as a reason in a proof? 
   a) Given  c) Definition 
   b) Prove  d) Postulate 
11. When can a theorem be cited as a “reason” for a proof? 
12. Based upon the hypothesis of a theorem, do the drawings 
   of different students have to be identical (same names for 
   vertices, etc.)? 

For each theorem stated in Exercises 13 to 18, make a 
Drawing. On the basis of your Drawing, write a Given and a 
Prove for the theorem.

13. If two lines are perpendicular, then these lines meet to form 
   a right angle. 
14. If two lines meet to form a right angle, then these lines are 
   perpendicular. 
15. If two angles are complementary to the same angle, then 
   these angles are congruent. 
16. If two angles are supplementary to the same angle, then 
   these angles are congruent. 
17. If two lines intersect, then the vertical angles formed are 
   congruent. 
18. Any two right angles are congruent. 

In Exercises 19 to 26, use the drawing in which \( \overline{AC} \) intersects 
\( \overline{DB} \) at point \( O \). 

19. If \( \angle 1 = 125^\circ \), find 
   \( \angle 2 \), \( \angle 3 \), and \( \angle 4 \). 
20. If \( \angle 2 = 47^\circ \), find \( \angle 1 \), \( \angle 3 \), and \( \angle 4 \). 
21. If \( \angle 1 = 3x + 10 \) and \( \angle 3 = 4x - 30 \), find \( x \) and 
   \( \angle 1 \). 
22. If \( \angle 2 = 6x + 8 \) and \( \angle 4 = 7x \), find \( x \) and \( \angle 2 \). 
23. If \( \angle 1 = 2x \) and \( \angle 2 = x \), find \( x \) and \( \angle 1 \). 
24. If \( \angle 2 = x + 15 \) and \( \angle 3 = 2x \), find \( x \) and \( \angle 2 \). 
25. If \( \angle 2 = \frac{1}{2} - 10 \) and \( \angle 3 = \frac{1}{2} + 40 \), find \( x \) and \( \angle 2 \). 
26. If \( \angle 1 = x + 20 \) and \( \angle 4 = \frac{1}{2} \), find \( x \) and \( \angle 4 \). 

In Exercises 27 to 35, complete the formal proof of 
each theorem.

27. If two angles are complementary to the same angle, then 
   these angles are congruent. 
   Given: \( \angle 1 \) is comp. to \( \angle 3 \) 
   \( \angle 2 \) is comp. to \( \angle 3 \) 
   Prove: \( \angle 1 \equiv \angle 2 \) 

### PROOF

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{BA} \perp \overline{BC} )</td>
<td>1. ?</td>
</tr>
<tr>
<td>2. ?</td>
<td>2. If two rays are ( \perp ), then they meet to form a rt. ( \angle )</td>
</tr>
<tr>
<td>3. ( \angle ABC = 90 )</td>
<td>3. ?</td>
</tr>
<tr>
<td>4. ( \angle ABC = \angle 1 + \angle 2 )</td>
<td>4. ?</td>
</tr>
<tr>
<td>5. ( \angle 1 + \angle 2 = 90 )</td>
<td>5. Substitution</td>
</tr>
<tr>
<td>6. ?</td>
<td>6. If the sum of the measures of two angles is 90, then the angles are complementary</td>
</tr>
</tbody>
</table>
32. If two line segments are congruent, then their midpoints separate these segments into four congruent segments.

Given:
\[ \overline{AB} \equiv \overline{DC} \]
\[ M \text{ is the midpoint of } \overline{AB} \]
\[ N \text{ is the midpoint of } \overline{DC} \]

Prove:
\[ AM \equiv MB \equiv DN \equiv NC \]

33. If two angles are congruent, then their bisectors separate these angles into four congruent angles.

Given:
\[ \angle ABC \equiv \angle EFG \]
\[ BD \text{ bisects } \angle ABC \]
\[ FH \text{ bisects } \angle EFG \]

Prove:
\[ \angle 1 \equiv \angle 2 \equiv \angle 3 \equiv \angle 4 \]

34. The bisectors of two adjacent supplementary angles form a right angle.

Given:
\[ \angle ABC \text{ is supp. to } \angle CBD \]
\[ BE \text{ bisects } \angle ABC \]
\[ BF \text{ bisects } \angle CBD \]

Prove:
\[ \angle EBF \text{ is a right angle} \]

35. The supplement of an acute angle is an obtuse angle.

(HINT: Use Exercise 28 of Section 1.6 as a guide.)

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**PERSPECTIVE ON HISTORY**

**THE DEVELOPMENT OF GEOMETRY**

One of the first written accounts of geometric knowledge appears in the Rhind papyrus, a collection of documents that date back to more than 1000 years before Christ. In this document, Ahmes (an Egyptian scribe) describes how north-south and east-west lines were redrawn following the overflow of the Nile River. Astronomy was used to lay out the north-south line. The rest was done by people known as “rope-fasteners.” By tying knots in a rope, it was possible to separate the rope into segments with lengths that were in the ratio 3 to 4 to 5. The knots were fastened at stakes in such a way that a right triangle would be formed. In Figure 1.69, the right angle is formed so that one side (of length 4, as shown) lies in the north-south line, and the second side (of length 3, as shown) lies in the east-west line.

The principle that was used by the rope-fasteners is known as the Pythagorean Theorem. However, we also know that the ancient Chinese were aware of this relationship. That is, the Pythagorean Theorem was known and applied many centuries before the time of Pythagoras (the Greek mathematician for whom the theorem is named).

Ahmes describes other facts of geometry that were known to the Egyptians. Perhaps the most impressive of these facts was that their approximation of \( \pi \) was 3.1604. To four decimal places of accuracy, we know today that the correct value of \( \pi \) is 3.1416.

Like the Egyptians, the Chinese treated geometry in a very practical way. In their constructions and designs, the Chinese used the rule (ruler), the square, the compass, and the level. Unlike the Egyptians and the Chinese, the Greeks formalized and expanded the knowledge base of geometry by pursuing it as an intellectual endeavor.

According to the Greek scribe Proclus (about 50 B.C.), Thales (625–547 B.C.) first established deductive proofs for several of the known theorems of geometry. Proclus also notes that it was Euclid (330–275 B.C.) who collected, summarized, ordered, and verified the vast quantity of knowledge of geometry in his time. Euclid’s work *Elements* was the first textbook of geometry. Much of what was found in *Elements* is the core knowledge of geometry and thus can be found in this textbook as well.