Chapter 1
Subtraction With Regrouping:
Approaches To Teaching A Topic

Scenario

Let’s spend some time thinking about one particular topic that you may work with when you teach, subtraction with regrouping. Look at these questions (\(\frac{25}{79}\) etc.). How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

When students first learn about subtraction, they learn to subtract each digit of the subtrahend from its counterpart in the minuend:

\[
\begin{array}{c}
75 \\
-12 \\
\hline
63
\end{array}
\]

To compute this, they simply subtract 2 from 5 and 1 from 7. However, this straightforward strategy does not work all the time. When a digit at a lower place value of the subtrahend is larger than its counterpart in the minuend (e.g., 22–14, 162–79), students cannot conduct the computation directly. To subtract 49 from 62, they need to learn subtraction with regrouping:

\[
\begin{array}{c}
56\underline{12} \\
-49 \\
\hline
13
\end{array}
\]

Subtraction, with or without regrouping, is a very early topic anyway. Is a deep understanding of mathematics necessary in order to teach it? Does such a simple topic even involve a deep understanding of mathematics? Would a teacher’s subject matter knowledge make any difference in his or her teaching, and eventually contribute to students’ learning? There is only one answer for all these questions: Yes. Even with such an elementary mathematical topic, the teachers displayed a wide range of subject matter knowledge, which suggests their students had a corresponding range of learning opportunities.

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THE U.S. TEACHERS’ APPROACH:
BORROWING VERSUS REGROUPING

Construing the Topic

When discussing their approach to teaching this topic, the U.S. teachers tended to begin with what they expected their students to learn. Nineteen of the 23 U.S. teachers (83%) focused on the procedure of computing. Ms. Fawn, a young teacher who had just finished her first year of teaching, gave a clear explanation of this procedure:

Whereas there is a number like 21–9, they would need to know that you cannot subtract 9 from 1, then in turn you have to borrow a 10 from the tens space, and when you borrow that 1, it equals 10, you cross out the 2 that you had, you turn it into a 10, you now have 11–9, you do that subtraction problem then you have the 1 left and you bring it down.

These teachers expected their students to learn how to carry out two particular steps: taking 1 ten from the tens place, and changing it into 10 ones. They described the “taking” step as borrowing. By noting the fact that “1 ten equals 10 ones,” they explained the step of “changing.” Here we can see the pedagogic insight of these teachers: Once their students can conduct these two key steps correctly, they will very likely be able to conduct the whole computation correctly.

The remaining four teachers, Tr. Bernadette, Tr. Bridget, Ms. Faith, and Ms. Fleur, however, expected their students to learn more than the computational procedure. They also expected their students to learn the mathematical rationale underlying the algorithm. Their approach emphasized two points: the regrouping underlying the “taking” step and the exchange underlying the “change” step. Tr. Bernadette, an experienced teacher, said:

They have to understand what the number 64 means… I would show that the number 64, and the number 5 tens and 14 ones, equal the 64. I would try to draw the comparison between that because when you are doing the regrouping it is not so much knowing the facts, it is the regrouping part that has to be understood. The regrouping right from the beginning.

Ms. Faith, another teacher at the end of her first year of teaching, indicated that students should understand that what happens in regrouping is the exchange within place values:

They have to understand how exchanges are done… the base 10 blocks when you reach a certain number—10, in base 10, in the ones column that is the same as, say, 10 ones or 1 ten… they have to get used to the idea that exchanges are made within place values and that it does not alter the value of the number…. Nothing happens to the actual value, but exchanges can be made.

What teachers expected students to know, however, was related to their own knowledge. The teachers who expected students merely to learn the procedure tended to have a proce-
dural understanding. To explain why one needs to “borrow” 1 ten from the tens place, these teachers said, “You can’t subtract a bigger number from a smaller number.” They interpreted the “taking” procedure as a matter of one number getting more value from another number, without mentioning that it is a within-number rearrangement:

You can’t subtract a bigger number from a smaller number… You must borrow from the next column because the next column has more in it. (Ms. Fay)

But if you do not have enough ones, you go over to your friend here who has plenty. (Tr. Brady)

“We can’t subtract a bigger number from a smaller one” is a false mathematical statement. Although second graders are not learning how to subtract a bigger number from a smaller number, it does not mean that in mathematical operations one cannot subtract a bigger number from a smaller number. In fact, young students will learn how to subtract a bigger number from a smaller number in the future. Although this advanced skill is not taught in second grade, a student’s future learning should not be confused by emphasizing a misconception.

To treat the two digits of the minuend as two friends, or two neighbors living next door to one another, is mathematically misleading in another way. It suggests that the two digits of the minuend are two independent numbers rather than two parts of one number.

Another misconception suggested by the “borrowing” explanation is that the value of a number does not have to remain constant in computation, but can be changed arbitrarily—if a number is “too small” and needs to be larger for some reason, it can just “borrow” a certain value from another number.

In contrast, the teachers who expected students to understand the rationale underlying the procedure showed that they themselves had a conceptual understanding of it. For example, Tr. Bernadette excluded any of the above misconceptions:

What do you think, the number, the number 64, can we take a number away, 46? Think about that. Does that make sense? If you have a number in the sixties can you take away a number in the forties? OK then, if that makes sense now, then 4 minus 6, are we able to do that? Here is 4, and I will visually show them 4. Take away 6, 1, 2, 3, 4. Not enough. OK, well what can we do? We can go to the other part of the number and take away what we can use, pull it away from the other side, pull it over to our side to help, to help the 4 become 14.

For Tr. Bernadette, the problem 64–46 was not, as suggested in the borrowing explanation, two separate processes of 4–6 and then 60–40. Rather, it was an entire process of “taking away a number in the forties from a number in the sixties.” Moreover, Tr. Bernadette thought that it was not that “you can’t subtract a bigger number from a smaller number,” rather, that the second graders “are not able to do that.” Finally, the solution was that “we go to the other part of the number” (italics added), and “pull it over to our side to help.” The difference between the phrases “other number” and “the other part of the number” is subtle, but the mathematical meanings conveyed are significantly different.

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Instructional Techniques: Manipulatives

Teachers’ knowledge of this topic was correlated not only with their expectations about student learning, but also with their teaching approaches. When discussing how they would teach the topic, all except one of the teachers referred to manipulatives. The most popular material was bundles of sticks (popsicle sticks, straws, or other kinds of sticks). Others were beans, money, base 10 blocks, pictures of objects, and games. The teachers said that by providing a “hands-on” experience, manipulatives would facilitate better learning than just “telling”—the way they had been taught.

A good vehicle, however, does not guarantee the right destination. The direction that students go with manipulatives depends largely on the steering of their teacher. The 23 teachers had different ideas that they wanted to get across by using manipulatives. A few teachers simply wanted students to have a “concrete” idea of subtraction. With the problem 52–25 for example, Tr. Belinda proposed “to have 52 kids line up and take 25 away and see what happens.” Ms. Florence reported that she would use beans as “dinosaur eggs” which might be interesting for students:

I would have them start some subtraction problems with maybe a picture of 23 things and tell them to cross out 17 and then count how many are left… I might have them do some things with dinosaur eggs or something that would sort of have a little more meaning to them. Maybe have them do some concrete subtraction with dinosaur eggs, maybe using beans as the dinosaur eggs or something.

Problems like 52–25 or 23–17 are problems of subtraction with regrouping. However, what students would learn from activities involving manipulatives like taking 25 students away from 52 or taking 17 dinosaur eggs away from 23 is not related to regrouping at all. On the contrary, the use of manipulatives removes the need to regroup. Tr. Barry, another experienced teacher in the procedurally directed group, mentioned using manipulatives to get across the idea that “you need to borrow something.” He said he would bring in quarters and let students change a quarter into two dimes and one nickel:

A good idea might be coins, using money because kids like money… . The idea of taking a quarter even, and changing it to two dimes and a nickel so you can borrow a dime, getting across that idea that you need to borrow something.

There are two difficulties with this idea. First of all, the mathematical problem in Tr. Barry’s representation was 25–10, which is not a subtraction with regrouping. Second, Tr. Barry confused borrowing in everyday life—borrowing a dime from a person who has a quarter—with the “borrowing” process in subtraction with regrouping—to regroup the minuend by rearranging within place values. In fact, Tr. Barry’s manipulative would not convey any conceptual understanding of the mathematical topic he was supposed to teach.

Most of the U.S. teachers said they would use manipulatives to help students understand the fact that 1 ten equals 10 ones. In their view, of the two key steps of the procedure, taking and changing, the latter is harder to carry out. Therefore, many teachers wanted to show this part visually or let students have a hands-on experience of the fact that 1 ten is actually 10 ones:
I would give students bundles of popsicle sticks that are wrapped in rubber bands, with 10 in each bundle. And then I’d write a problem on the board, and I would have bundles of sticks, as well, and I would first show them how I would break it apart (italics added), to go through the problem, and then see if they could manage doing the same thing, and then, maybe, after a lot of practice, maybe giving each pair of students a different subtraction problem, and then they could, you know, demonstrate, or give us their answer. Or, have them make up a problem using sticks, breaking them apart and go through it. (Ms. Fiona)

What Ms. Fiona reported was a typical method used by many teachers. Obviously, it is related more to subtraction with regrouping than the methods described by Ms. Florence and Tr. Barry. However, it still appears procedurally focused. Following the teacher’s demonstration, students would practice how to break a bundle of 10 sticks apart and see how it would work in the subtraction problems. Although Ms. Fiona described the computational procedure clearly, she did not describe the underlying mathematical concept at all.

Scholars have noted that in order to promote mathematical understanding, it is necessary that teachers help to make connections between manipulatives and mathematical ideas explicit (Ball, 1992; Driscoll, 1981; Hiebert, 1984; Resnick, 1982; Schram, Nemser, & Ball, 1989). In fact, not every teacher is able to make such a connection. It seems that only the teachers who have a clear understanding of the mathematical ideas included in the topic might be able to play this role. Ms. Faith, the beginning teacher with a conceptual understanding of the topic, said that by “relying heavily upon manipulatives” she would help students to understand “how each one of these bundles is 10, it is 1 ten or 10 ones;” to know that “5 tens and 3 ones is the same as 4 tens and 13 ones;” to learn “the idea of equivalent exchange,” and to talk about “the relationship with the numbers”:

What I would do, from that point, is show how each one of these bundles is 10, it is 1 ten or 10 ones. I would make sure that was clear. And what would happen if we undid this little rubber band and put 10 over here, how many ones would we have? And to get to the next step, I would show that now you have 1, 2, 3, 4 tens and 13 ones and then subtract in that fashion … I would say to the child so you are telling me that we have not added anything or subtracted anything to the 53, right? Right… Five tens and 3 ones is the same as 4 tens and 13 ones, and what happens when you take 25 from that?

Unlike the other teachers who used manipulatives to illustrate the computational procedure, Ms. Faith used them to represent the mathematical concept underlying the procedure. The only reason that Ms. Faith’s use of manipulatives could help her students “further” than that of other teachers was that she understood the mathematical topic in a deeper way than others. Using a similar method, teachers with different views of the subject matter would lead students to different understandings of mathematics.

Ms. Y. was in the middle of her second year of teaching. Her explanation was a variant of Ms. Fawn’s. She focused on the specific steps of the algorithm and did not show any interest in its rationale. The proportion of Chinese teachers who held such procedurally directed ideas, however, was substantially smaller than that of the U.S. teachers (14% vs. 83%). Figure 1.1 shows the teachers’ different understandings of the topic.

Most of the Chinese teachers focused on regrouping. However, in contrast to the U.S. teachers, about 35% of the Chinese teachers described more than one way of regrouping. These teachers not only addressed the rationale for the standard algorithm, but also discussed other ways to solve the problem that were not mentioned by the U.S. teachers. Let’s first take a look at the catchphrase of the Chinese teachers: decomposing a unit of higher value.

“Decomposing a unit of higher value [tui yi *]” is a term in Chinese traditional arithmetic reckonings by the abacus. Each wire on an abacus represents a certain place value. The value of each bead on the abacus depends on the position of the wire that carries the bead. The farther to the left a wire is located on the abacus, the larger the place value it represents. Therefore, the values of beads on left wires are always greater than those on right wires.

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FIG. 1.1. Teachers’ understanding of subtraction with regrouping.

When subtracting with regrouping on the abacus, one needs to “take” a bead on a left wire and change it into 10 or powers of 10 beads on a wire to the right. This is called “decomposing a unit of higher value.”

* The Chinese characters for this and other Chinese words appear in Appendix Fig. A.2.
Eighty-six percent of the Chinese teachers described the “taking” step in the algorithm as a process of “decomposing a unit of higher value.” Instead of saying that “you borrow 1 ten from the tens place,” they said that “you decompose 1 ten.”

The reason that one cannot compute 21–9 directly lies in the form of the number 21. In the decimal system, the numbers are composed according to the rate of 10. Given that a number gets 10 units at a certain place value (e.g., ones place or tens place), the 10 units should be organized into 1 unit of the next higher place value (e.g., tens place or hundreds place). Theoretically, no more than 9 “scattered” (uncomposed) units exist in the decimal number system. Now we want to subtract 9 scattered ones units from 21. The latter only has 1 ones unit. The solution, then, is to decompose a unit of higher value, a 10, and subtract 9 individual ones units from the recomposed 21.

During the interviews, the teachers tended to discuss the idea of “decomposing a higher valued unit” in connection to addition with carrying—“composing a unit of higher value [jin yi].” In describing how she would teach this topic, Tr. L., an experienced teacher who teaches first through third grades, said:

I would start with a problem of straightforward subtraction, like 43−22=?. After they solve it, I would change the problem into 43−27=? How does the new problem differ from the first one? What will happen when we compute the second problem? They will soon find that 7 is bigger than 3 and we do not have enough ones. Then I would say, OK, today we don’t have enough ones. But sometimes we have too many ones. You must remember that last week when we did addition with carrying we had too many ones. What did we do at that time? They will say we composed them into tens. So when we have too many ones, we compose them into tens, what can we do when we don’t have enough ones? We may decompose a 10 back to ones. If we decompose a 10 in 40, what will happen? We will have enough ones. In this way I will introduce the concept of “decomposing a unit of higher value into 10 units of a lower value.”

Some teachers indicated that the term “decomposing” suggests its relationship with the concept of “composing.”

How come there are not enough ones in 53 to subtract 6? Fifty-three is obviously bigger than 6. Where are the ones in 53? Students will say that the other ones in 53 have been composed into tens. Then I will ask them what we can do to get enough ones to subtract 7. I expect that they will come up with the idea of decomposing a 10. Otherwise, I will propose it. (Tr. P.)

In China, as in the United States, the term “borrowing” used to be a traditional metaphor in subtraction. Ms. S., a third-grade teacher in her second year of teaching, explained why she thought that the concept of “decomposing a higher value unit” made more sense than the metaphor of borrowing:

Some of my students may have learned from their parents that you “borrow one unit from the tens and regard it as 10 ones [jie yi dang shi].” I will explain to them that we are not borrowing a 10, but decomposing a 10. “Borrowing” can’t explain why you can take a 10 to the ones place. But “decomposing” can. When you say decomposing, it implies that the digits in higher places are actually composed of those at lower places. They are exchangeable. The term “borrowing” does not mean the composing-decomposing process at all. “Borrowing one unit and turning it into 10” sounds arbitrary. My students may ask me how can we borrow from the tens place? If we borrow something, we should return it later on. How and what are we going to return? Moreover, when borrowing we should get a person who would like to lend to us. How about if the tens place does not want to lend to the ones place? You will not be able to answer these questions that students may ask.

To construe the “taking” step as decomposing a unit of higher value reflects an even more comprehensive understanding than the explanation that draws on “regrouping.” Although the rationale of the algorithm is regrouping the minuend, regrouping, however, is a mathematical approach that is not confined to subtraction. It is fundamental to a variety of mathematical computations. There are various ways of regrouping. For example, when conducting addition with carrying, the sum at a certain place may be more than 10 units. Then we regroup it by composing the units into one, or more, unit(s) of a higher place value. Again, doing multidigit multiplication we regroup the multiplier into groups of the same place value (e.g., in computing 57×39, one regroups 39 into 30+9 and conducts the computation as 57×30+57×9). In fact, each of the four arithmetical operations applies some kind of regrouping. Therefore, explaining the “taking” procedure in terms of “regrouping” is correct, because regrouping is less relevant to the topic of subtraction than “decomposing a unit of higher value.” The former fails to indicate the specific form of regrouping occurring in the subtraction.

Moreover, in using the concept of decomposing a higher value unit, the subtraction procedure is explained in a way that shows its connection with the operation of addition. It not only provides more conceptual support for the learning of subtraction, but reinforces students’ previous learning.

The Rate of Composing a Higher Value Unit. With the concept of “decomposing 1 ten into 10 ones,” the conceptually directed Chinese teachers had actually explained both the “taking” and the “changing” steps in the algorithm. However, many of them further discussed the “changing” aspect of the procedure. About half of them, like the U.S. teachers in the “regrouping” group, emphasized that 1 ten is composed of 10 ones and can be decomposed into 10 ones. The other half, however, referred to a more basic mathematical idea—the rate for composing a higher value unit [jin lu]—as a concept that students need to know before learning regrouping, and that should be reinforced throughout teaching.

These teachers asserted that students should have a clear idea about “the rate for composing a higher value unit” so that they can better understand why a higher value unit is decomposed into 10, or powers of 10, lower value units. This understanding, according to these teachers, will facilitate students’ future learning.

1 This aspect has also been observed by other scholars. Stigler and Perry (1988a) reported that Chinese teachers emphasize “the composition and decomposition of numbers into groups of ten.”

2 Early versions of modern Chinese arithmetic textbooks used the term “subtraction with borrowing” translated from the West. During the past few decades, the textbooks have used instead “subtraction with decomposing.”
What is the rate for composing a higher value unit? The answer is simple: 10. Ask students how many ones there are in a 10, or ask them what the rate for composing a higher value unit is, their answers will be the same: 10. However, the effect of the two questions on their learning is not the same. When you remind students that 1 ten equals 10 ones, you tell them the fact that is used in the procedure. And, this somehow confines them to the fact. When you require them to think about the rate for composing a higher value unit, you lead them to a theory that explains the fact as well as the procedure. Such an understanding is more powerful than a specific fact. It can be applied to more situations. Once they realize that the rate of composing a higher value unit, 10, is the reason why we decompose a ten into 10 ones, they will apply it to other situations. You don’t need to remind them again that 1 hundred equals 10 tens when in the future they learn subtraction with three-digit numbers. They will be able to figure it out on their own.

Ms. N. had taught lower grade classes at an elementary school in a rural area for three years. She said:

To discuss the rate for composing a higher value unit here is not only helpful for them to deal with subtraction of multidigit numbers, but also other more complicated versions of problems. To decompose a ten into 10 ones or to decompose a hundred into 10 tens is to decompose 1 unit into 10 units of the next lower value. But sometimes we need to decompose one unit into one hundred, one thousand or even more units of lower value. For example, to compute 302–17, we need to decompose one hundred into 100 ones. Again, conducting the subtraction 10,005–206, we need to decompose one unit into ten-thousand lower-valued units. If our students are limited to the fact that 1 ten equals 10 ones, they may feel confused when facing these problems. But if at the beginning of learning, they are exposed to the rate for composing a higher value unit, they may be able to deduce the solutions of these new problems. Or at least they have a key to solving the problems.

Teachers like Tr. Mao and Ms. N. shared a keen foresight in students’ learning. Their approach to teaching subtraction with two-digit numbers foresaw the related skills needed for subtraction with multidigit numbers. Multidigit subtraction includes problems of decomposing a hundred into tens, or decomposing a thousand into hundreds. It may also include problems of decomposing a unit not into 10, but into a power of ten lower units, such as decomposing a thousand into 100 tens, etc. This “foresight,” obviously, is based on these teachers’ thorough understanding of this topic.

When learning addition with carrying, students of these teachers are exposed to the idea of the rate for composing a higher value unit. When teaching subtraction these teachers lead their students to revisit the idea from another perspective—the perspective of decomposing a unit. This visit is certainly an enhancement of their earlier learning of the basic idea.

Compared with the concept of exchanging 1 ten and 10 ones, the idea of the rate for composing a higher value unit reaches a more profound layer of mathematical understanding. Bruner (1960/1977), in The Process of Education, said, “The more fundamental or basic is the idea he has learned, almost by definition, the greater will be its breadth of applicability to new problems” (p. 18). Indeed, the rate for composing a higher value unit is a basic idea of the number system. Connecting the “changing” step with the idea of composing a unit in the number system reflects these teachers’ insight into the basic ideas underneath the facts, and their capacity to embody a fundamental idea of the discipline in a single fact.

Multiple Ways of Regrouping. The previous discussion has been confined to the standard algorithm for solving subtraction problems. The algorithm has a procedure for regrouping the minuend in a certain way, for example, 53 is regrouped as 40 and 13. Although none of the U.S. teachers went beyond this standard way, some Chinese teachers did. These teachers pointed out that the algorithm is not the only correct way to conduct the subtraction. There are also other ways that will work. The standard way works best in most cases, but not in all cases. Around the principle of “decomposing a higher value unit,” the teachers discussed various ways of regrouping:

Actually there are several ways of grouping and regrouping that we can use to think about the problem 53–26. First of all, we can regroup 53 in this way:

\[
\begin{array}{c}
\text{53} \\
/ \\
\text{40} \\
\text{13}
\end{array}
\]

In this way, we can subtract 6 from 13, 20 from 40, and get 27. This makes sense. However, we also may want to regroup 53 in another way:

\[
\begin{array}{c}
\text{53} \\
/ \\
\text{40} \\
\text{10} \\
\text{3}
\end{array}
\]

We subtract 6 from 10 and get 4, add the 4 to 3 and get 7, subtract 20 from 40, add the 7 to 20 and get 27. The advantage of this second way of regrouping is that it is easier to subtract 6 from 10 than from 13. The addition included in this procedure does not involve carrying so it is simple too. There is still another way to regroup. We may want to regroup the subtrahend 26 as:

\[
\begin{array}{c}
\text{26} \\
/ \\
\text{20} \\
\text{33}
\end{array}
\]

We first subtract one 3 from 53 and get 50. Then we subtract the other 3 from 50 and get 47. Finally we subtract 20 from 47 and get 27. (Tr. C)

The teachers referred to three main ways of regrouping. One was the standard way: decompose a unit at a higher value place into units at a lower value place, combine them with the original units at the lower place, and then subtract.

Another way was to regroup the minuend into three parts, rather than two parts, before subtracting. In other words, leave the unit split from the tens place, instead of combining it
with the units at the ones place. Then subtract the subtrahend’s digit at the ones place from the split unit. Finally, combine the difference with the minuend’s units at the ones place. Although the additional part of the number seems to create some complexity, this computation is even easier than in the standard way. One simply needs to subtract the minuend from 10, rather than from a number larger than 10.

Subtraction with the third way of regrouping may be even easier. First, split from the ones place of the subtrahend the same number that is at the ones place of the minuend. Next, subtract the split number from the minuend, which makes the ones place of the minuend zero. Then subtract the rest of the subtrahend from the minuend that is now composed of whole tens.

The second and third ways are actually used frequently in daily life. These approaches are also usually more acceptable to young children because of their limited capacity in mathematics. In addition to describing these alternative ways of regrouping, the Chinese teachers also compared them—describing the situations when these methods may make the computation easier. Some teachers said that the second way of regrouping is used more often when the lowest placed digit of the subtrahend is substantially larger than that of the minuend. For example, 52–7, or 63–9. These problems are easy to solve if one first subtracts 7 from 50 and adds 2 to the first difference 43, or first subtracts 9 from 60 and adds 3 to the first difference 51. For in this kind of problem, the subtrahends are usually close to 10.

The third way is particularly easy when the value of the digits of the minuend and of the subtrahend at the lower place are close to each other. For example, 47–8, or 95–7. It is easy to subtract 7 from 47 and then subtract 1 from the first difference 40, or, to subtract 5 from 95 and then subtract 2 from the first difference 90.

Despite the number of ways to subtract, the standard way is still the best one for most problems, in particular, those that are more complicated. Tr. Li, a recognized teacher, described what happens in her classroom when she teaches subtraction:

We start with the problems of a two-digit number minus a one-digit number, such as 34–6. I put the problem on the board and ask students to solve the problem on their own, either with bundles of sticks or other learning aids, or even with nothing, just thinking. After a few minutes, they finish. I have them report to the class what they did. They might report a variety of ways. One student might say “34–6, 4 is not enough to subtract 6. But I can take off 4 first, get 30. Then I still need to take 2 off. Because 6=4+2. I subtract 2 from 30 and get 28.” Another student who worked with sticks might say, “When I saw that I did not have enough separate sticks, I broke 1 bundle. I got 10 sticks and I put 6 of them away. There were 4 left. I put the 4 sticks with the original 4 sticks together and got 8. I still have another two bundles of 10s, putting the sticks left all together I had 28.” Some students, usually fewer than the first two kinds, might report, “The two ways they used are fine, but I have another way to solve the problem. We have learned how to compute 14–8, 14–9, why don’t we use that knowledge. So, in my mind I computed the problem in a simple way. I regrouped 34 into 20 and 14. Then I subtracted 6 from 14 and got 8. Of course I did not forget the 20, so I got 28.” I put all the ways students reported on the board and label them with numbers, the first way, the second way, etc. Then I invite students to compare: Which way do you think is the easiest? Which way do you think is most reasonable? Sometimes they don’t agree with each other. Sometimes they don’t agree that the standard way I am to teach is the easiest way. Especially for those who are not proficient and comfortable with problems of subtraction within 20, such as 13–7, 15–8, etc., they tend to think that the standard way is more difficult.

Students may actually come upon various ways of regrouping if they try to solve the problems by themselves. This was reported by other teachers as well. To lead a thoughtful discussion once students have expressed all their ideas, a teacher needs a thorough comprehension of this topic. He or she should know these various solutions of the problem, know how and why students came up with them, know the relationship between the non-standard ways and the standard way, and know the single conception underlying all the different ways. Tr. G., a second-grade teacher in her early thirties, concluded, after describing the various ways her students might solve a problem using manipulatives:

I would lead the class to discover that there is one process underlying all various ways of subtraction: to un-binding one bundle. This would bring them to understand the concept of decomposing a ten, which plays the key role in the computation.

It is important for a teacher to know the standard algorithm as well as alternative versions. It is also important for a teacher to know why a certain method is accepted as the standard one, while the other ways can still play a significant role in the approach to the knowledge underlying the algorithm. With a broad perspective in comparing and contrasting the various ways of regrouping in subtraction, the concept underlying the procedure is revealed thoroughly. Supported by a comprehensive understanding of the conception, these teachers were able to show a flexibility in dealing with the nonstandard methods not included in textbooks.

Another interesting feature of the Chinese teachers’ interviews was that they tended to address connections among mathematical topics. For example, most of the Chinese teachers mentioned the issue of “subtraction within 20” as the conceptual, as well as procedural, “foundation” for subtraction with regrouping.

They said that the idea of regrouping in subtraction, to decompose a higher value unit into lower value units, is developed through learning three levels of problems:

The first level includes problems with minuends between 10 and 20, like 15–7, 16–8, etc. At this level, students learn the concept of decomposing a 10 and the skill derived from it. They learn that by decomposing a 10, they will be able to subtract one-digit numbers from “teen-numbers” with one’s digit smaller than the subtrahend. This step is critical because before that, subtraction was straightforward—one subtracted small one-digit num-

Knowledge Package and Its Key Pieces

Another interesting feature of the Chinese teachers’ interviews was that they tended to address connections among mathematical topics. For example, most of the Chinese teachers mentioned the issue of “subtraction within 20” as the conceptual, as well as procedural, “foundation” for subtraction with regrouping.

They said that the idea of regrouping in subtraction, to decompose a higher value unit into lower value units, is developed through learning three levels of problems:

The first level includes problems with minuends between 10 and 20, like 15–7, 16–8, etc. At this level, students learn the concept of decomposing a 10 and the skill derived from it. They learn that by decomposing a 10, they will be able to subtract one-digit numbers from “teen-numbers” with one’s digit smaller than the subtrahend. This step is critical because before that, subtraction was straightforward—one subtracted small one-digit num-

3 By the term “subtraction within 20,” Chinese teachers mean subtraction with regrouping with minuends between 10 and 20, such as 12–6 or 15–7. By the term “addition within 20,” Chinese teachers mean addition with carrying where the sum is between 10 and 20, such as 7+8 or 9+9.
bers from larger one-digit numbers or from “teen-numbers” with ones digits larger than
the subtrahend.4 The conception and the skill learned at this level will support regrouping
procedures at the other levels.

The second level includes problems with minuends between 19 and 100, like 53–25,
72–48, etc. At the second level, the ten to be decomposed is combined with several tens.
The new idea is to split it from the other tens.

The third level includes problems with larger minuends, that is, minuends with three or
more digits. The new idea in the third level is successive decomposition. When the next
higher place in a minuend is a zero, one has to decompose a unit from further than the next
higher place. The problems involve decomposing more than once, and sometimes even
twice. For example, in the problem 203–15, working at the ones place, one needs to
decompose 1 hundred into 10 tens, and moreover, decompose 1 ten into 10 ones.

According to the Chinese teachers, the basic idea of subtraction with regrouping de-velops through the three levels. However, the conceptual “seed” and the basic skill throughout
all levels of the problems occur as early as the first level—subtraction within 20.

Here is a very interesting difference in understanding between the two countries. In
the United States, problems like “5+7=12” or “12–7=5” are considered “basic arithmetic facts”
for students simply to memorize. In China, however, they are considered problems of
“addition with composing and subtraction with decomposing within 20.”5 The learning of
“addition with composing and subtraction with decomposing within 20” is the first occa-
sion when students must draw on previous learning, in this case their skill of composing
and decomposing a 10 is significantly embedded.

Tr. Sun was in her late thirties. She had taught for eighteen years at elementary schools
in several cities. She even questioned my interview question, thinking that it was not rel-

vant enough:

The topic you raised was subtraction with regrouping. But the problems you showed
me here, which all have minuends bigger than 20 and less than 100, are only one kind
of problem in learning this topic. In fact, this is not the crucial kind of problem for
learning this topic. It is hard for me to talk about how to teach the topic only drawing
on the approach to these problems.

The Chinese number-word system may contribute to Chinese teachers’ particular attention to
composing and decomposing a 10. In Chinese, all the “teen-numbers” have the form “ten, a one-
digit number.” For example, eleven is “ten-one,” twelve is “ten-two,” and so on. (Twenty is “two
tens,” thirty is “three tens,” and so on. Twenty-one is called “two tens-one,” twenty-two is “two
tens-two” and so on.) Therefore, “decomposing the 10” tends to be an obvious solution for the
problem of “How one can subtract 5 from ten-two?”

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tens-two” and so on.) Therefore, “decomposing the 10” tends to be an obvious solution for the
problem of “How one can subtract 5 from ten-two?”

5 In China, addition with carrying is called “addition with composing” and subtraction with
regrouping is called “subtraction with decomposing.” “Addition with composing and subtraction
with decomposing within 20” is taught during the second semester of first grade.

6 In Chinese elementary mathematics textbooks, before the section on “addition with composing
and subtraction with decomposing within 20” there is a section on the composition of a 10. Until
students reach the section of addition and subtraction within 20, however, the mathematical mean-
ing of composing and decomposing a 10 is not clear to them.
I asked Tr. Chen further about “knowledge package” and its size and the components. He responded:

There is not a firm, rigid, or single right way to “pack” knowledge. It is all up to one’s own viewpoint. Different teachers, in different contexts, or the same teacher with different students, may “pack” knowledge in different ways. But the point is that you should see a knowledge “package” when you are teaching a piece of knowledge. And you should know the role of the present knowledge in that package. You have to know that the knowledge you are teaching is supported by which ideas or procedures, so your teaching is going to rely on, reinforce, and elaborate the learning of these ideas. When you are teaching an important idea that will support other procedures, you should devote particular efforts to make sure that your students understand the idea very well and are able to carry out the procedure proficiently.

Most of the Chinese teachers, like Tr. Chen, talked about a group of pieces of knowledge rather than a single piece of knowledge. The following network sketch was drawn based on their discussion of subtraction with regrouping. As Tr. Chen said, to “pack” knowledge—to see mathematical topics group-by-group rather than piece-by-piece—is a way of thinking. The teachers’ opinions of what and how many knowledge pieces should be included in the “package” differed somewhat. What they shared were the principles of how to “pack” the knowledge and what the “key” pieces were. Figure 1.2 illustrates the main ideas the Chinese teachers use when they “pack” the knowledge pieces related to subtraction with regrouping. The rectangle represents the topic I raised in the interview. The ellipses represent the related knowledge pieces. The shaded ellipses represent the key pieces of knowledge. An arrow from one topic to another indicates that the first topic supports the second, thus, according to the teachers, should occur prior to the second in teaching.⁷

At the middle of the figure there is a sequence of four topics: “addition and subtraction within 10,” “addition and subtraction within 20,” “subtraction with regrouping of numbers between 20 and 100,” and “subtraction with regrouping of large numbers.” According to the Chinese teachers, the concept and procedure of subtraction with regrouping develops step-by-step through this sequence, from a primary and simple form to a complex and advanced form. The topic of “addition and subtraction within 20” is considered the key piece of the sequence to which the teachers devote most effort in the whole process of teaching subtraction with regrouping. They believe that the concept as well as the computational skill introduced with the topic “addition and subtraction within 20” constitute the basis for later learning of more advanced forms of subtraction with regrouping. Therefore, it will provide powerful support to students’ later learning of subtraction, both conceptually and procedurally.

Besides the central sequence, the knowledge package also contains a few other topics. Directly connected to one or more links in the sequence, directly or indirectly, encircle the sequence. During their interviews, some teachers discussed a “sub-sequence” of “circle”—from “the composition of 10” to “addition without carrying” to “subtraction without regrouping.” We that with a change of perspective, for example, if our topic is how to subtraction without regrouping, subsequence might become the central sequence in the teachers’ knowledge package. One topic in the “circle,” “composing and decomposing a higher value unit,” is considered to be another key piece in the package because it is the core concept underlying the subtraction algorithm.

The purpose of a teacher in organizing knowledge in such a package is to promote a solid learning of a certain topic. It is obvious that all the items in the subtraction package are related to the learning of this topic, either supporting, or supported by it. Some items, for example, subtraction without regrouping, are included mainly to provide a procedural support. Other items, for example, composing and decomposing a higher value unit, are considered mainly as a conceptual support. Still others, for example, the concept of inverse operation, were referred to as conceptual support as well as procedural support.⁸ Individual teacher’s networks varied according to the size and the specific items included. However, the relationships between the items and core items were common.

**Manipulatives and Other Teaching Approaches**

Although mentioned less frequently than among the U.S. teachers, manipulatives were also a strategy often reported by the Chinese teachers. What differed was that most Chinese teachers said that they would have a class discussion following the use of manipulatives. In these

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⁷ During interviews the Chinese teachers often commented that the relationship is two-way: first learning of a basic topic supports the learning of a more advanced topic, but the learning of a basic topic is also reinforced by the latter. Because the focus of this study is teaching, I did not put two-way arrows in the knowledge package figures.

⁸ A few Chinese teachers mentioned that they would remind students to “think about addition when doing subtraction” to facilitate their learning.
discussions students may report, display, explain, and argue for their own solutions. Through the discussions, “the explicit construction of links between understood actions on the objects and related symbol procedures” claimed by Hiebert (1984, p. 509) would be established.

Leading a discussion after using manipulatives, however, demands more breadth and depth in a teacher’s subject matter knowledge. Through the manipulatives, various issues may be raised by the students. If a teacher does not know very well the different ways to solve a problem, how can he or she lead a discussion about the different ways students report to the class?

Sometimes a class discussion must deal with more intriguing problems that cannot be solved in one lesson. Ms. S. reported a discussion in her class that started at the beginning of the school year and concluded at the end:

Last fall when my students worked on this kind of problems with manipulatives, we found a problem. We found that the manipulative procedure was not the same as we do it on paper with columns. Say we are doing the problem 35–18. With manipulatives we start from the higher value place. We take the 10 in 18 first and then take the 8 out. With columns we start with the ones place, subtracting the 8 first. The way with manipulatives, in fact, is the way we do most subtraction in our everyday life. When we think about how much change we will get after paying 2 Yuans9 for something that costs 1 Yuan and 63 cents, we first subtract 1 Yuan and then 60 cents and then the 3 cents. But with the standard method in columns we do it the opposite way. We subtract 3 cents first, then 60 cents, and finally 1 Yuan. From the perspective of students’ life experience, the way they learn in school seems to be more complex and makes less sense. We tried it on the board to see what would happen if we started from the higher place. We found that starting with the tens place we will first get a difference of 2 at the tens place:

\[
\begin{array}{c}
35 \\
-18 \\
\hline
2
\end{array}
\]

Then when we worked at the ones place, it happened that we had to change the difference at the tens place that we just got:

\[
\begin{array}{c}
35 \\
-18 \\
\hline
2
\end{array}
\]

But if we started from the ones place, this trouble could be avoided. We would get a final difference directly. Yet this explanation only solved half of the problem—why with columns we need to start at the lower place. Students were still not convinced that they had to learn the standard way, since they did not see an obvious advantage in using the standard way. I suggested that we save the puzzle, probably we would come back to the issue sometime later on. At the end of the school year, we worked on subtraction with decomposing larger numbers. I raised the question again for a discussion. My students soon found that with larger numbers, the standard way is much easier with most problems. Then they agreed that the standard way is worthwhile to learn….

If Ms. S’s knowledge had been limited to how to conduct the computational procedure, it would be hard to imagine that she could lead her students to such a mathematical understanding.

**DISCUSSION**

**Making Connections: Consciously Versus Unconsciously**

Certainly a teacher’s subject matter knowledge of mathematics differs from that of a non-teaching person. Special features of a teacher’s subject matter knowledge are derived from the task of promoting student learning. To facilitate learning, teachers tend to make explicit the connections between and among mathematical topics that remain tacit for non-teachers. In discussions of teaching subtraction with regrouping, teachers tended to make two kinds of connections. First, they tended to connect the topic with one or a few related procedural topics, usually those of lower status such as the procedure of subtraction without grouping and the fact that 1 ten equals 10 ones. Obviously, these are the basis for subtraction with regrouping. Second, the teachers tended to connect the procedure with an explanation. This also reinforces students’ learning—by giving a reason for “taking” and “changing,” the teacher provides more information to support the learning of the algorithm.

When they were asked what they thought pupils would need to understand or be able to do before learning subtraction with regrouping, all the teachers presented their own “knowledge package” including both kinds of connections. One difference, however, was that some teachers showed a definite consciousness of the connections, while others did not. This difference was associated with significant differences in teachers’ subject knowledge. The teachers who tended to “pack” knowledge consciously could describe the elements they included in the package. In addition, they were clearly aware of the structure of the network, and the status of each element in it.

On the other hand, those teachers who packed knowledge unconsciously were vague and uncertain of the elements and the structure of the network. The knowledge packages in their minds were underdeveloped. Indeed, although connecting a topic that is to be taught to related topics may be a spontaneous intention of any teaching person, a fully developed and well-organized knowledge package about a topic is a result of deliberate study.

**Models of Teachers’ Knowledge of Subtraction:**

**Procedural Understanding Versus Conceptual Understanding**

Most knowledge packages that the teachers described during interviews contained the same kinds of elements—those providing procedural support and those providing explanations. Teachers with conceptual understanding and teachers with only procedural understanding, however, had differently organized knowledge packages.

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9 Yuan is a unit for Chinese money. One Yuan is 100 cents.
**A Model of Procedural Understanding of Subtraction With Regrouping.** The knowledge packages of the teachers with only a procedural understanding of subtraction contained few elements. Most of these elements were procedural topics directly related to the algorithm of subtraction with regrouping. A brief explanation was usually included, but it was not a real mathematical explanation. For example, when a teacher told his or her students that the rationale of the algorithm is just like their mother goes to a neighbor to borrow some sugar, this arbitrary explanation doesn’t contain any real mathematical meaning. Some teachers explained that because the digit at the ones column of the minuend is smaller than that of the subtrahend, the former should “borrow” a ten from the tens column and turn it into ten ones. This was not a real mathematical explanation either. As discussed earlier in this chapter, some explanations were even mathematically problematic. The understanding of these teachers appeared conceptual, but in fact was too faulty and fragmented to promote students’ conceptual learning.

Figure 1.3 illustrates a knowledge package of a teacher with procedural understanding. The top rectangle represents procedural knowledge of the algorithm. The two ellipses represent related procedural topics. The trapezoid underneath the rectangle represents pseudo-conceptual understanding.

Eighty-three percent of the U.S. teachers’ and 14% of the Chinese teachers’ knowledge about subtraction with regrouping fell into this pattern. Their understanding of the topic contained a few procedural topics and a pseudo-conceptual understanding. They made very few connections among mathematical topics and no mathematical arguments were involved in their explanations.

**FIG. 1.3.** Procedural understanding of a topic.

**A Model of Conceptual Understanding of Subtraction.** The knowledge of the teachers with a conceptual understanding of subtraction was differently considered and organized. Three kinds of mathematical knowledge are included in a fully developed and well-organized knowledge package of conceptual understanding: procedural topics, conceptual topics, and basic principles of the subject. **Procedural topics** are included to support the procedural learning, as well as the conceptual learning of the topic. For example, proficiency in composing and decomposing a 10 is such a procedural topic. Many Chinese teachers referred to it as a significant support for learning the addition and subtraction within 20, procedurally as well as conceptually. **Conceptual topics** are included mainly for a thorough understanding of the rationale underlying the algorithm. However, the teachers believed that conceptual topics also played an important role in promoting procedural proficiency. For instance, some teachers thought that a comprehensive understanding of the concept of regrouping helped students to choose an easy method of subtraction.

Some teachers’ knowledge packages included **basic principles**, for example, the concept of the **rate of composing a higher value unit** and the concept of **inverse operations**. The rate of composing a higher value unit is a basic principle of understanding numeral systems. This concept is not only related to students’ learning of subtraction with regrouping of large numbers when successive decomposition is necessary, but will also be related to students’ later learning of the binary system—a completely different numeral system. Moreover, by revealing a principle of numeral systems, the concept will deepen one’s understanding of the whole subject.

The concept of inverse operations is one of the main principles that underlie the relationships among the operations of mathematics. Though this concept is related to the learning of subtraction with its inverse operation, addition, it also supports the learning of other inverse operations in mathematics, such as multiplication and division, squaring and taking square roots, cubing and taking cube roots, raising to the $n$th roots, etc.

These two general principles are examples of what Bruner (1960/1977) called “the structure of the subject.” Bruner said, “Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure, in short, is to learn how things are related” (p. 7).

Indeed, the teachers who tend to include “simple but powerful” basic ideas of the subject in their teaching would not only promote a conceptual learning in the present, but also prepare their students to relate their present learning to future learning.

A well-developed conceptual understanding of a topic also includes understanding of another dimension of structure of the subject—attitudes toward mathematics. Again, Bruner said, “Mastery of fundamental ideas of a field involves not only the grasping of general principles, but the development of an attitude toward learning and inquiry, toward guessing and hunches, toward the possibility of solving problems on one’s own” (p. 20).

The teachers did not give any examples of attitudes toward mathematics in their knowledge packages. A few teachers, however, displayed their knowledge of general attitudes. Their discussions of the conventional and alternative ways of regrouping displayed an attitude of the subject—that of approaching a mathematical issue from various perspectives. Teachers’ descriptions of encouraging students to present their own ways of doing subtraction with regrouping and leading them to a discussion of these ways showed the teachers’ own attitudes toward mathematical inquiry. In addition, the teachers’ intention of providing mathematical proof after raising an issue, their confidence and capacity in discussing the topic in a mathematical way, and their intention to promote such discussion among their students are all examples of general attitudes. In fact, though they were not explicitly included as particular items in any teacher’s knowledge package, basic attitudes of mathematics have a strong influence on conceptual understanding of mathematics. As I note in later chapters, most of the specific topics mentioned in this chapter do not appear in discussions of multidigit multiplication, division by fractions, and area and perimeter. The attitudes the teachers presented in this chapter, however, will accompany us through the other data chapters and the remainder of the book.
Figure 1.2 displayed how a well-developed knowledge package for subtraction with regrouping was organized. Figure 1.4 illustrates a model of conceptual understanding of a topic. The uppermost gray rectangle represents procedural understanding of the topic. The central gray trapezoid represents conceptual understanding of the topic. It is supported by a few procedural topics (while ellipses), regular conceptual topics (light gray ellipses), basic of mathematics (dark ellipses as principles and dot-lined ellipses as basic attitudes of mathematics). The bottom rectangle represents the structure of mathematics.

An authentic conceptual understanding is supported by mathematical arguments. For example, the U.S. teachers who held a conceptual understanding elaborated the “regrouping” aspect of the operation. Many Chinese teachers explained that the main idea of the algorithm is “decomposing a higher value unit”. Both explanations are based on mathematical arguments and reflected the teachers’ conceptual understanding of the procedural topic.

The conceptual understanding of subtraction with regrouping, however, does not share “only one correct answer.” There are various versions of conceptual explanations. For example, Teacher A might discuss the concept of decomposing higher value unit. Teacher B might discuss the concept of decomposing relative to the concept of composing. Teacher C might introduce the concept of rate of composing a higher value unit. Teacher D might present the concept of regrouping using the regrouping suggested by the algorithm. Teacher E might present several ways of regrouping to elaborate the concept. All these teachers have authentic conceptual understandings. However, the breadth and depth of their understandings are not the same. The shading on the trapezoid is intended to display this feature of conceptual understanding.

We know very little about the quality and features of teachers’ conceptual understanding. One thing that may be true is that the mathematical power of a concept depends on its relationship with other concepts. The closer a concept is to the structure of the subject, the more relationships it may have with other topics. If a teacher introduces a basic principle of the subject to explain the rationale of the procedure of subtraction with regrouping, he or she endows that explanation with a strong mathematical power.

Seventeen percent of the U.S. teachers and 86% of the Chinese teachers demonstrated a conceptual understanding of the topic. Among these teachers, the Chinese teachers presented a more sophisticated knowledge than their U.S. counterparts.

### Relationship Between Subject Matter Knowledge and Teaching Method: Can the Use of Manipulatives Compensate for Subject Matter Knowledge Deficiency?

Compared with subject matter knowledge, other aspects of teaching usually receive more attention, perhaps because they seem to affect students more directly. In thinking of how to teach a topic a major concern will be what approach to use. During their interviews most teachers said that they would use manipulatives. However, the way in which manipulatives would be used depended on the mathematical understanding of the teacher using them. The 23 U.S. teachers did not have the same learning goals. Some wanted students to have a “concrete” idea of subtraction, some wanted students to understand that 1 ten equals 10 ones, and one wanted students to learn the idea of equivalent exchange. Those who wanted students to have a concrete idea of subtraction described uses of manipulatives that eliminated the need to regroup. Those who wanted students to understand that 1 ten equals 10 ones described a procedure with manipulatives that students could use for computation. The teacher who wanted students to learn the idea of equivalent exchange described how she would use manipulatives to illustrate the concept underlying the procedure. In contrast to the U.S. teachers, the Chinese teachers said they would have a class discussion following the use of manipulatives in which students would report, display, explain, and argue for their solutions.

In activities involving manipulatives, and particularly in the discussions described by the Chinese teachers, students may raise questions that would lead to a deeper understanding of mathematics. The realization of the learning potential of such questions may still largely rely on the quality of the teacher’s subject matter knowledge.

### SUMMARY

Subtraction with regrouping is so elementary that it is hard to imagine that teachers might not possess adequate knowledge of this topic. However, the interviews in this chapter revealed this was the case for some teachers. Eighty-three percent of the US teachers and 14% of the Chinese teachers displayed only procedural knowledge of the topic. Their understanding was limited to surface aspects of the algorithm—the taking and changing steps. This limitation in their knowledge confined their expectations of student learning as well as their capacity to promote conceptual learning in the classroom.

This chapter also revealed different layers of conceptual understanding of subtraction with regrouping. Some U.S. teachers explained the procedure as regrouping the minuend and said that during instruction they would point out the “exchanging” aspect underly-
ing the “changing” step. Most of the Chinese teachers explained the regrouping used in subtraction computations as decomposing a higher value unit. More than one third of the Chinese teachers discussed nonstandard methods of regrouping and relationships between standard and nonstandard methods.

Teachers with different understandings of subtraction with regrouping had different instructional goals. Although many teachers mentioned using manipulatives as a teaching approach, the uses they described, which would largely decide the quality of learning in class, depended on what they thought students should learn. In contrast with U.S. teachers, most Chinese teachers said that after students had used manipulatives they would have a class discussion—a teaching strategy that requires more breadth and depth of a teacher's subject matter knowledge.