II #6

Lemma: If \( p \) divides \( ab \), and if \( p \) does not divide \( a \), then \( p \) must divide \( b \).

Theorem: Show that if \( p \) is prime, and \( k \in \mathbb{Z} \) such that \( 1 \leq k \leq p - 1 \), then \( p | \binom{p}{k} \).

Proof:
Firstly, \( \binom{p}{k} \) is a positive integer because of its combinatorial meaning.

\[
\binom{p}{k} = \frac{p!}{(p-k)!k!} = \frac{p(p-1)(p-2) \ldots (p-(k-1))!}{k!} \\
k! \binom{p}{k} = p(p-1)(p-2) \ldots (p-(k-1))!
\]

Since \( p \) is a factor of the RHS, it must be a factor of \( k! \binom{p}{k} \). Because \( k < p \), it follow that \( k! \) does not contain a copy of \( p \), and since \( p \) is prime, \( k! \) also cannot contain the factors of \( p \) (since \( p \) has no factors). Therefore, \( p \) does not divide \( k! \). So \( p \) must divide \( \binom{p}{k} \).

IV. Set Theory

(a) (vi) \( B \subseteq A \cup B \)

Proof: To prove a statement involving \( \subseteq \) we need to show that anytime an arbitrary element \( x \) is in the left side, it must also be in the right side. So, let \( x \in B \), then by “addition” property of propositional logic, it’s true that \( (x \in B) \lor (x \in A) \). So, \( x \in A \cup B \).

VI. Induction

(a) (i)

Using the division algorithm, if we divide \( n \) by 3, we get \( n = 3q + r \), it follows that \( 0 \leq r < 3 \). This implies that \( n \equiv 0 \pmod{3} \) or \( n \equiv 1 \pmod{3} \), or \( n \equiv 2 \pmod{3} \)

Explanation for the formula for \( n \) choose \( k \):

\[
\binom{n}{k} = \frac{n(n-1)(n-(k-1))!}{1} = \frac{n!}{(n-k)!k!}
\]

A, B, C, D, E

ABCDE, ACDEB, ADBCE, AEBCD, BACDE, ...

\( 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! \)

A,B,C,D,E

ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEB, AED,
BAC, BCA,...

\[ P(5, 3) = 5 \cdot 4 \cdot 3 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!} = 60 \]

\[ \binom{5}{3} = C(5, 3) = \frac{60}{6} = 10 \]

(VII) (b)

**Theorem:** The Divisibility Criterion for 9: If the sum of the digits of a number is divisible by 9, then the number is divisible by 9.

**Proof:** Let’s restrict it to a 4 digit number. Let \( n \) be a 4 digit number, such that 9 divides the sum of its digits. First, \( n \) can be written as \( n = 1000a + 100b + 10c + d \) where \( a, b, c, d \) are its digits. \( n = 999a + 99b + 9c + (a + b + c + d) \)

By hypothesis, \( 9 \mid (a + b + c + d) \). Furthermore \( 9 \mid (999a + 99b + 9c) \), so 9 divides their sum which is \( n \).

(VII) (e)

Suppose \( m \) and \( n \) are perfect squares. So, there exist \( t, s \) such that \( m = t^2 \) and \( n = s^2 \).

\( mn = t^2s^2 = (ts)^2 \), so \( mn \) is also a perfect square.

(m) **Lemma:** \( a \equiv b \pmod{p} \rightarrow a^n \equiv b^n \pmod{p} \)

Remember Fermat’s little theorem: Assuming \( p \) does not divide \( b \), \( b^{p-1} \equiv 1 \pmod{p} \).

\[ 11^{3-1} \equiv 1 \pmod{3} \]

\[ 11^{300} = (11^2)^{150} \equiv 1^{150} \equiv 1 \pmod{3} \]

\[ 11^{301} = 11^{300} \cdot 11 \equiv 1 \cdot 11 \equiv 11 \equiv 2 \pmod{3} \]