Forced Oscillations and Resonance

In Section 2.4 we derived the differential equation

\[ mx'' + cx' + kx = F(t) \]  

(1)

that governs the one-dimensional motion of a mass \( m \) that is attached to a spring (with constant \( k \)) and a dashpot (with constant \( c \)) and is also acted on by an external force \( F(t) \). Machines with rotating components commonly involve mass-spring systems (or their equivalents) in which the external force is simple harmonic:

\[ F(t) = F_0 \cos \omega t \quad \text{or} \quad F(t) = F_0 \sin \omega t, \]  

(2)

where the constant \( F_0 \) is the amplitude of the periodic force and \( \omega \) is its circular frequency.

For an example of how a rotating machine component can provide a simple harmonic force, consider the cart with a rotating vertical flywheel shown in Fig. 2.6.1. The cart has mass \( m - m_0 \), not including the flywheel of mass \( m_0 \). The centroid of the flywheel is off center at a distance \( a \) from its center, and its angular speed is \( \omega \) radians per second. The cart is attached to a spring (with constant \( k \)) as shown. Assume that the centroid of the cart itself is directly beneath the center of the flywheel, and denote by \( x(t) \) its displacement from its equilibrium position (where the spring is unstretched). Figure 2.6.1 helps us to see that the displacement \( \bar{x} \) of the centroid of the combined cart plus flywheel is given by

\[
\bar{x} = \frac{(m - m_0)x + m_0(x + a \cos \omega t)}{m} = x + \frac{m_0 a}{m} \cos \omega t.
\]

Let us ignore friction and apply Newton’s second law \( m\ddot{x} = -kx \), because the force exerted by the spring is \( -kx \). We substitute for \( \bar{x} \) in the last equation to obtain

\[
mx'' - m_0 a \omega^2 \cos \omega t = -kx;
\]

that is,

\[
mx'' + kx = m_0 a \omega^2 \cos \omega t.
\]

Thus the cart with its rotating flywheel acts like a mass on a spring under the influence of a simple harmonic external force with amplitude \( F_0 = m_0 a \omega^2 \). Such a system is a reasonable model of a front-loading washing machine with the clothes being washed loaded off center. This illustrates the practical importance of analyzing solutions of Eq. (1) with external forces as in (2).

Undamped Forced Oscillations

To study undamped oscillations under the influence of the external force \( F(t) = F_0 \cos \omega t \), we set \( c = 0 \) in Eq. (1), and thereby begin with the equation

\[ mx'' + kx = F_0 \cos \omega t \]  

(4)

whose complementary function is \( x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \). Here

\[
\omega_0 = \sqrt{\frac{k}{m}}
\]
(as in Eq. (9) of Section 2.4) is the (circular) natural frequency of the mass–spring system. The fact that the angle \( \omega_0 t \) is measured in (dimensionless) radians reminds us that if \( t \) is measured in seconds (s), then \( \omega_0 \) is measured in radians per second—that is, in inverse seconds (s\(^{-1}\)). Also recall from Eq. (14) in Section 2.4 that division of a circular frequency \( \omega \) by the number \( 2\pi \) of radians in a cycle gives the corresponding (ordinary) frequency \( v = \omega/2\pi \) in Hz (hertz = cycles per second).

Let us assume initially that the external and natural frequencies are unequal: \( \omega \neq \omega_0 \). We substitute \( x_p = A \cos \omega t \) in Eq. (4) to find a particular solution. (No sine term is needed in \( x_p \) because there is no term involving \( x' \) on the left-hand side in Eq. (4).) This gives

\[
-m\omega^2 A \cos \omega t + k A \cos \omega t = F_0 \cos \omega t,
\]

so

\[
A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2},
\]

and thus

\[
x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.
\]

Therefore, the general solution \( x = x_c + x_p \) is given by

\[
x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t,
\]

where the constants \( c_1 \) and \( c_2 \) are determined by the initial values \( x(0) \) and \( x'(0) \). Equivalently, as in Eq. (12) of Section 2.4, we can rewrite Eq. (7) as

\[
x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t,
\]

so we see that the resulting motion is a superposition of two oscillations, one with natural circular frequency \( \omega_0 \), the other with the frequency \( \omega \) of the external force.

---

**Example 1**

Suppose that \( m = 1 \), \( k = 9 \), \( F_0 = 80 \), and \( \omega = 5 \), so the differential equation in (4) is

\[
x'' + 9x = 80 \cos 5t.
\]

Find \( x(t) \) if \( x(0) = x'(0) = 0 \).

**Solution**

Here the natural frequency \( \omega_0 = 3 \) and the frequency \( \omega = 5 \) of the external force are unequal, as in the preceding discussion. First we substitute \( x_p = A \cos 5t \) in the differential equation and find that \(-25A + 9A = 80\), so that \( A = -5 \). Thus a particular solution is

\[
x_p(t) = -5 \cos 5t.
\]

The complementary function is \( x_c = c_1 \cos 3t + c_2 \sin 3t \), so the general solution of the given nonhomogeneous equation is

\[
x(t) = c_1 \cos 3t + c_2 \sin 3t - 5 \cos 5t,
\]

with derivative

\[
x'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t + 25 \sin 5t.
\]
The initial conditions \( x(0) = 0 \) and \( x'(0) = 0 \) now yield \( c_1 = 5 \) and \( c_2 = 0 \), so the desired particular solution is

\[
x(t) = 5 \cos 3t - 5 \cos 5t.
\]

As indicated in Fig. 2.6.2, the period of \( x(t) \) is the least common multiple \( 2\pi \) of the periods \( 2\pi/3 \) and \( 2\pi/5 \) of the two cosine terms.


Beats

If we impose the initial conditions \( x(0) = x'(0) = 0 \) on the solution in (7), we find that

\[
c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0,
\]

so the particular solution is

\[
x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t).
\]

(9)

The trigonometric identity \( 2 \sin A \sin B = \cos(A - B) - \cos(A + B) \), applied with \( A = \frac{1}{2}(\omega_0 + \omega)t \) and \( B = \frac{1}{2}(\omega_0 - \omega)t \), enables us to rewrite Eq. (9) in the form

\[
x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{1}{2}(\omega_0 - \omega)t \sin \frac{1}{2}(\omega_0 + \omega)t.
\]

(10)

Suppose now that \( \omega \approx \omega_0 \), so that \( \omega_0 + \omega \) is very large in comparison with \( |\omega_0 - \omega| \). Then \( \sin \frac{1}{2}(\omega_0 + \omega)t \) is a rapidly varying function, whereas \( \sin \frac{1}{2}(\omega_0 - \omega)t \) is a slowly varying function. We may therefore interpret Eq. (10) as a rapid oscillation with circular frequency \( \frac{1}{2}(\omega_0 + \omega) \),

\[
x(t) = A(t) \sin \frac{1}{2}(\omega_0 + \omega)t,
\]

but with a slowly varying amplitude

\[
A(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{1}{2}(\omega_0 - \omega)t.
\]

With \( m = 0.1 \), \( F_0 = 50 \), \( \omega_0 = 55 \), and \( \omega = 45 \), Eq. (10) gives

\[
x(t) = \sin 5t \sin 50t.
\]

Figure 2.6.3 shows the corresponding oscillation of frequency \( \frac{1}{2}(\omega_0 + \omega) = 50 \) that is “modulated” by the amplitude function \( A(t) = \sin 5t \) of frequency \( \frac{1}{2}(\omega_0 - \omega) = 5 \).

A rapid oscillation with a (comparatively) slowly varying periodic amplitude exhibits the phenomenon of beats. For example, if two horns not exactly attuned to one another simultaneously play their middle C, one at \( \omega_0/(2\pi) = 258 \) Hz and the other at \( \omega/(2\pi) = 254 \) Hz, then one hears a beat—an audible variation in the amplitude of the combined sound—with a frequency of

\[
\frac{(\omega_0 - \omega)/2}{2\pi} = \frac{258 - 254}{2} = 2 \text{ (Hz)}.
\]
Resonance

Looking at Eq. (6), we see that the amplitude $A$ of $x_p$ is large when the natural and external frequencies $\omega_0$ and $\omega$ are approximately equal. It is sometimes useful to rewrite Eq. (5) in the form

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0}{k} \cdot \left(1 - \frac{(\omega/\omega_0)^2}{1 - (\omega/\omega_0)^2}\right) = \pm \rho \frac{F_0}{k},$$

(11)

where $F_0/k$ is the static displacement of a spring with constant $k$ due to a constant force $F_0$, and the amplification factor $\rho$ is defined to be

$$\rho = \frac{1}{\left|1 - (\omega/\omega_0)^2\right|}.$$  

(12)

It is clear that $\rho \to +\infty$ as $\omega \to \omega_0$. This is the phenomenon of resonance—the increase without bound (as $\omega \to \omega_0$) in the amplitude of oscillations of an undamped system with natural frequency $\omega_0$ in response to an external force with frequency $\omega \approx \omega_0$.

We have been assuming that $\omega \neq \omega_0$. What sort of catastrophe should one expect if $\omega$ and $\omega_0$ are precisely equal? Then Eq. (4), upon division of each term by $m$, becomes

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t.$$  

(13)

Because $\cos \omega_0 t$ is a term of the complementary function, the method of undetermined coefficients calls for us to try

$$x_p(t) = t(A \cos \omega_0 t + B \sin \omega_0 t).$$

We substitute this in Eq. (13) and thereby find that $A = 0$ and $B = F_0/(2m\omega_0)$. Hence the particular solution is

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$  

(14)

The graph of $x_p(t)$ in Fig. 2.6.4 (in which $m = 1$, $F_0 = 100$, and $\omega_0 = 50$) shows vividly how the amplitude of the oscillation theoretically would increase without bound in this case of pure resonance, $\omega = \omega_0$. We may interpret this phenomenon as reinforcement of the natural vibrations of the system by externally impressed vibrations at the same frequency.

**Example 3**

Suppose that $m = 5$ kg and $k = 500$ N/m in the cart with the flywheel of Fig. 2.6.1. Then the natural frequency is $\omega_0 = \sqrt{k/m} = 10$ rad/s; that is, $10/(2\pi) \approx 1.59$ Hz. We would therefore expect oscillations of very large amplitude to occur if the flywheel revolves at about $(1.59)(60) \approx 95$ revolutions per minute (rpm).

In practice, a mechanical system with very little damping can be destroyed by resonance vibrations. A spectacular example can occur when a column of soldiers marches in step over a bridge. Any complicated structure such as a bridge has many natural frequencies of vibration. If the frequency of the soldiers’ cadence is approximately equal to one of the natural frequencies of the structure, then—just as in our simple example of a mass on a spring—resonance will occur. Indeed, the resulting
resonance vibrations can be of such large amplitude that the bridge will collapse. This has actually happened—for example, the collapse of Broughton Bridge near Manchester, England, in 1831—and it is the reason for the now-standard practice of breaking cadence when crossing a bridge. Resonance may have been involved in the 1981 Kansas City disaster in which a hotel balcony (called a skywalk) collapsed with dancers on it. The collapse of a building in an earthquake is sometimes due to resonance vibrations caused by the ground oscillating at one of the natural frequencies of the structure; this happened to many buildings in the Mexico City earthquake of September 19, 1985. On occasion an airplane has crashed because of resonant wing oscillations caused by vibrations of the engines. It is reported that for some of the first commercial jet aircraft, the natural frequency of the vertical vibrations of the airplane during turbulence was almost exactly that of the massspring system consisting of the pilot’s head (mass) and spine (spring). Resonance occurred, causing pilots to have difficulty in reading the instruments. Large modern commercial jets have different natural frequencies, so that this resonance problem no longer occurs.

Modeling Mechanical Systems

The avoidance of destructive resonance vibrations is an ever-present consideration in the design of mechanical structures and systems of all types. Often the most important step in determining the natural frequency of vibration of a system is the formulation of its differential equation. In addition to Newton’s law \( F = ma \), the principle of conservation of energy is sometimes useful for this purpose (as in the derivation of the pendulum equation in Section 2.4). The following kinetic and potential energy formulas are often useful.

1. Kinetic energy: \( T = \frac{1}{2}mv^2 \) for translation of a mass \( m \) with velocity \( v \);
2. Kinetic energy: \( T = \frac{1}{2}I\omega^2 \) for rotation of a body of a moment of inertia \( I \) with angular velocity \( \omega \);
3. Potential energy: \( V = \frac{1}{2}kx^2 \) for a spring with constant \( k \) stretched or compressed a distance \( x \);
4. Potential energy: \( V = mgh \) for the gravitational potential energy of a mass \( m \) at height \( h \) above the reference level (the level at which \( V = 0 \)), provided that \( g \) may be regarded as essentially constant.

Find the natural frequency of a mass \( m \) on a spring with constant \( k \) if, instead of sliding without friction, it is a uniform disk of radius \( a \) that rolls without slipping, as shown in Fig 2.6.5.

Solution

With the preceding notation, the principle of conservation of energy gives
\[
\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = E
\]
where \( E \) is a constant (the total mechanical energy of the system). We note that \( v = a\omega \) and recall that \( I = ma^2/2 \) for a uniform circular disk. Then we may simplify the last equation to
\[
\frac{3}{4}mv^2 + \frac{1}{2}kx^2 = E.
\]
Because the right-hand side of this equation is constant, differentiation with respect to \( t \) (with \( v = x' \) and \( v' = x'' \)) now gives
\[
\frac{3}{2}mx'' + kxx' = 0.
\]
We divide each term by \( \frac{3}{2}mx' \) to obtain
\[
x'' + \frac{2k}{3m}x = 0.
\]

Thus the natural frequency of horizontal back-and-forth oscillation of our rolling disk is \( \sqrt{\frac{2k}{3m}} \), which is \( \sqrt{\frac{2}{3}} \approx 0.8165 \) times the familiar natural frequency \( \sqrt{\frac{k}{m}} \) of a mass on a spring that is sliding without friction rather than rolling without sliding. It is interesting (and perhaps surprising) that this natural frequency does not depend on the radius of the disk. It could be either a dime or a large disk with a radius of one meter (but of the same mass).

**Example 5**

Suppose that a car oscillates vertically as if it were a mass \( m = 800 \text{ kg} \) on a single spring (with constant \( k = 7 \times 10^4 \text{ N/m} \)), attached to a single dashpot (with constant \( c = 3000 \text{ N·s/m} \)). Suppose that this car with the dashpot disconnected is driven along a washboard road surface with an amplitude of 5 cm and a wavelength of \( L = 10 \text{ m} \) (Fig. 2.6.6). At what car speed will resonance vibrations occur?

**Solution**

We think of the car as a unicycle, as pictured in Fig. 2.6.7. Let \( x(t) \) denote the upward displacement of the mass \( m \) from its equilibrium position; we ignore the force of gravity, because it merely displaces the equilibrium position as in Problem 9 of Section 2.4. We write the equation of the road surface as
\[
y = a \cos \frac{2\pi s}{L} \quad (a = 0.05 \text{ m}, \; L = 10 \text{ m}).
\] (15)

When the car is in motion, the spring is stretched by the amount \( x - y \), so Newton’s second law, \( F = ma \), gives
\[
mx'' = -k(x - y);
\]
that is,
\[
mx'' + kx = ky
\] (16)

If the velocity of the car is \( v \), then \( s = vt \) in Eq. (15), so Eq. (16) takes the form
\[
mx'' + kx = k \cos \frac{2\pi vt}{L}.
\] (16′)
This is the differential equation that governs the vertical oscillations of the car. In comparing it with Eq. (4), we see that we have forced oscillations with circular frequency \( \omega = 2\pi v/L \). Resonance will occur when \( \omega = \omega_0 = \sqrt{k/m} \). We use our numerical data to find the speed of the car at resonance:

\[
v = \frac{L}{2\pi} \sqrt{\frac{k}{m}} = \frac{10}{2\pi} \sqrt{\frac{7 \times 10^4}{800}} \approx 14.89 \text{ (m/s)};
\]

that is, about 33.3 mi/h (using the conversion factor of 2.237 mi/h per m/s).

Damped Forced Oscillations

In real physical systems there is always some damping, from frictional effects if nothing else. The complementary function \( x_c \) of the equation

\[
mx'' + cx' + kx = F_0 \cos \omega t
\]

is given by Eq. (19), (20), or (21) of Section 2.4, depending on whether \( c > c_{cr} = \sqrt{4km} \), \( c = c_{cr} \), or \( c < c_{cr} \). The specific form is not important here. What is important is that, in any case, these formulas show that \( x_c(t) \to 0 \) as \( t \to +\infty \). Thus \( x_c \) is a transient solution of Eq. (17)—one that dies out with the passage of time, leaving only the particular solution \( x_p \).

The method of undetermined coefficients indicates that we should substitute

\[
x(t) = A \cos \omega t + B \sin \omega t
\]

in Eq. (17). When we do this, collect terms, and equate coefficients of \( \cos \omega t \) and \( \sin \omega t \), we obtain the two equations

\[
(k - m\omega^2)A + c\omega B = F_0, \quad -c\omega A + (k - m\omega^2)B = 0
\]

that we solve without difficulty for

\[
A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.
\]

If we write

\[
A \cos \omega t + B \sin \omega t = C(\cos \omega t \cos \alpha + \sin \omega t \sin \alpha) = C \cos(\omega t - \alpha)
\]

as usual, we see that the resulting steady periodic oscillation

\[
x_p(t) = C \cos(\omega t - \alpha)
\]

has amplitude

\[
C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.
\]

Now (19) implies that \( \sin \alpha = B/C > 0 \), so it follows that the phase angle \( \alpha \) lies in the first or second quadrant. Thus

\[
\tan \alpha = \frac{B}{A} = \frac{c\omega}{k - m\omega^2} \quad \text{with} \quad 0 < \alpha < \pi,
\]
so
\[
\alpha = \begin{cases} 
\tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k > m\omega^2, \\
\pi + \tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k < m\omega^2
\end{cases}
\]
(whereas \(\alpha = \pi/2\) if \(k = m\omega^2\)).

Note that if \(c > 0\), then the “forced amplitude”—defined as a function \(C(\omega)\) by (21)—always remains finite, in contrast with the case of resonance in the undamped case when the forcing frequency \(\omega\) equals the critical frequency \(\omega_0 = \sqrt{k/m}\). But the forced amplitude may attain a maximum for some value of \(\omega\), in which case we speak of practical resonance. To see if and when practical resonance occurs, we need only graph \(C\) as a function of \(\omega\) and look for a global maximum. It can be shown (Problem 27) that \(C\) is a steadily decreasing function of \(\omega\) if \(c \geq \sqrt{2km}\). But if \(c < \sqrt{2km}\), then the amplitude of \(C\) attains a maximum value—and so practical resonance occurs—at some value of \(\omega\) less than \(\omega_0\), and then approaches zero as \(\omega \to +\infty\). It follows that an underdamped system typically will undergo forced oscillations whose amplitude is

- Large if \(\omega\) is close to the critical resonance frequency;
- Close to \(F_0/k\) if \(\omega\) is very small;
- Very small if \(\omega\) is very large.

**Example 6**

Find the transient motion and steady periodic oscillations of a damped mass-and-spring system with \(m = 1\), \(c = 2\), and \(k = 26\) under the influence of an external force \(F(t) = 82 \cos 4t\) with \(x(0) = 6\) and \(x'(0) = 0\). Also investigate the possibility of practical resonance for this system.

**Solution**

The resulting motion \(x(t) = x_D(t) + x_{sp}(t)\) of the mass satisfies the initial value problem

\[
x'' + 2x' + 26x = 82 \cos 4t; \quad x(0) = 6, \quad x'(0) = 0.
\]

(23)

Instead of applying the general formulas derived earlier in this section, it is better in a concrete problem to work it directly. The roots of the characteristic equation

\[r^2 + 2r + 26 = (r + 1)^2 + 25 = 0\]

are \(r = -1 \pm 5i\), so the complementary function is

\(x_c(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)\).

When we substitute the trial solution

\(x(t) = A \cos 4t + B \sin 4t\)

in the given equation, collect like terms, and equate coefficients of \(\cos 4t\) and \(\sin 4t\), we get the equations

\[10A + 8B = 82,\]
\[-8A + 10B = 0\]
with solution $A = 5, B = 4$. Hence the general solution of the equation in (23) is

$$x(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t) + 5 \cos 4t + 4 \sin 4t.$$ 

At this point we impose the initial conditions $x(0) = 6, x'(0) = 0$ and find that $c_1 = 1$ and $c_2 = -3$. Therefore, the transient motion and the steady periodic oscillation of the mass are given by

$$x_{tr}(t) = e^{-t}(\cos 5t - 3 \sin 5t)$$

and

$$x_{sp}(t) = 5 \cos 4t + 4 \sin 4t = \sqrt{41} \left( \frac{5}{\sqrt{41}} \cos 4t + \frac{4}{\sqrt{41}} \sin 4t \right)$$

$$= \sqrt{41} \cos (4t - \alpha)$$

where $\alpha = \tan^{-1} \left( \frac{4}{5} \right) \approx 0.6747$.

Figure 2.6.8 shows graphs of the solution $x(t) = x_{tr}(t) + x_{sp}(t)$ of the initial value problem

$$x'' + 2x' + 26x = 82 \cos 4t, \quad x(0) = x_0, \quad x'(0) = 0 \quad (24)$$

for the different values $x_0 = -20, -10, 0, 10, \text{ and } 20$ of the initial position. Here we see clearly what it means for the transient solution $x_{tr}(t)$ to "die out with the passage of time," leaving only the steady periodic motion $x_{sp}(t)$. Indeed, because $x_{tr}(t) \to 0$ exponentially, within a very few cycles the full solution $x(t)$ and the steady periodic solution $x_{sp}(t)$ are virtually indistinguishable (whatever the initial position $x_0$).

![FIGURE 2.6.8. Solutions of the initial value problem in (24) with $x_0 = -20, -10, 0, 10, \text{ and } 20$.]
In each of Problems 7 through 10, find the steady periodic solution $x_{sp}(t)$ of the given equation $mx'' + cx' + kx = F(t)$ with periodic forcing function $F(t)$ of frequency $\omega$. Then graph $x_{sp}(t)$ together with (for comparison) the adjusted forcing function $F_0(t) = F(t)/m\omega_0$. In each of Problems 11 through 14, find and plot both the steady periodic solution $x_{sp}(t) = C \cos(\omega t - \alpha)$ of the given differential equation and the transient solution $x_{tr}(t)$ that satisfies the given initial conditions.

1. $x'' + 9x = 10 \cos 2t; \quad x(0) = x'(0) = 0$
2. $x'' + 4x = 5 \sin 3t; \quad x(0) = x'(0) = 0$
3. $x'' + 100x = 225 \cos 5t + 300 \sin 5t; \quad x(0) = 375, x'(0) = 0$
4. $x'' + 25x = 90 \cos 4t; \quad x(0) = 0, x'(0) = 90$
5. $mx'' + kx = F_0 \cos \omega t$ with $\omega \neq \omega_0; \quad x(0) = x_0, x'(0) = 0$
6. $mx'' + kx = F_0 \cos \omega_0 t$ with $\omega = \omega_0; \quad x(0) = 0, x'(0) = v_0$

In each of Problems 15 through 18, give the parameters for a forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady periodic forced oscillations with frequency $\omega$. Sketch the graph of $C(\omega)$ and find the practical resonance frequency $\omega$ (if any).

15. $m = 1, c = 2, k = 2, F_0 = 2$
16. $m = 1, c = 4, k = 5, F_0 = 10$
17. $m = 1, c = 6, k = 45, F_0 = 50$
18. $m = 1, c = 10, k = 650, F_0 = 100$

19. A mass weighing 100 lb (mass $m = 3.125$ slugs in fps units) is attached to the end of a spring that is stretched 1 in. by a force of 100 lb. A force $F_0 \cos \omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillations occur? Neglect damping.

20. A front-loading washing machine is mounted on a thick rubber pad that acts like a spring; the weight $W = mg$ (with $g = 9.8$ m/s$^2$) of the machine depresses the pad exactly 0.5 cm. When its rotor spins at $\omega$ radians per second, the rotor exerts a vertical force $F_0 \cos \omega t$ newtons on the machine. At what speed (in revolutions per minute) will resonance vibrations occur? Neglect friction.

21. Figure 2.6.10 shows a mass $m$ on the end of a pendulum (of length $L$) also attached to a horizontal spring (with constant $k$). Assume small oscillations of $m$ so that the spring remains essentially horizontal and neglect damping. Find the natural circular frequency $\omega_0$ of motion of the mass in terms of $L, k, m$, and the gravitational constant $g$. To investigate the possibility of practical resonance in the given system, we substitute the values $m = 1, c = 2, k = 26$ in (21) and find that the forced amplitude at frequency $\omega$ is

$$C(\omega) = \frac{82}{\sqrt{676 - 48\omega^2 + \omega^4}}.$$

The graph of $C(\omega)$ is shown in Fig. 2.6.9. The maximum amplitude occurs when

$$C'(\omega) = \frac{-41(4\omega^3 - 96\omega)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = \frac{-164\omega(\omega^2 - 24)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = 0.$$

Thus practical resonance occurs when the external frequency is $\omega = \sqrt{24}$ (a bit less than the mass-and-spring’s undamped critical frequency of $\omega_0 = \sqrt{k/m} = \sqrt{26}$).
22. A mass \( m \) hangs on the end of a cord around a pulley of radius \( a \) and moment of inertia \( I \), as shown in Fig. 2.6.11. The rim of the pulley is attached to a spring (with constant \( k \)). Assume small oscillations so that the spring remains essentially horizontal and neglect friction. Find the natural circular frequency of the system in terms of \( m \), \( a \), \( k \), \( I \), and \( g \).

23. A building consists of two floors. The first floor is attached rigidly to the ground, and the second floor is of mass \( m = 1000 \) slugs (fps units) and weighs 16 tons (32,000 lb). The elastic frame of the building behaves as a spring that resists horizontal displacements of the second floor; it requires a horizontal force of 5 tons to displace the second floor a distance of 1 ft. Assume that in an earthquake the ground oscillates horizontally with amplitude \( A_0 \) and circular frequency \( \omega_0 \), resulting in an external horizontal force \( F(t) = m A_0 \omega_0^2 \sin \omega_0 t \) on the second floor. (a) What is the natural frequency (in hertz) of oscillations of the second floor? (b) If the ground undergoes one oscillation every 2.25 s with an amplitude of 3 in., what is the amplitude of the resulting forced oscillations of the second floor?

24. A mass on a spring without damping is acted on by the external force \( F(t) = F_0 \cos^3 \omega_0 t \). Show that there are two values of \( \omega \) for which resonance occurs, and find both.

25. Derive the steady periodic solution of

\[ mx'' + cx' + kx = F_0 \sin \omega t. \]

In particular, show that it is what one would expect—the same as the formula in (20) with the same values of \( C \) and \( \omega \), except with \( \sin(\omega t - \alpha) \) in place of \( \cos(\omega t - \alpha) \).

26. Given the differential equation

\[ mx'' + cx' + kx = E_0 \cos \omega t + F_0 \sin \omega t \]

—with both cosine and sine forcing terms—derive the steady periodic solution

\[ x_{sp}(t) = \frac{\sqrt{E_0^2 + F_0^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \alpha - \beta), \]

where \( \alpha \) is defined in Eq. (22) and \( \beta = \tan^{-1}(F_0/E_0) \). (Suggestion: Add the steady periodic solutions separately corresponding to \( E_0 \cos \omega t \) and \( F_0 \sin \omega t \) (see Problem 25).)

27. According to Eq. (21), the amplitude of forced steady periodic oscillations for the system \( mx'' + cx' + kx = F_0 \cos \omega t \) is given by

\[ C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}. \]

(a) If \( c \geq c_{cr}/\sqrt{2} \), where \( c_{cr} = \sqrt{4km} \), show that \( C \) steadily decreases as \( \omega \) increases. (b) If \( c < c_{cr}/\sqrt{2} \), show that \( C \) attains a maximum value (practical resonance) when

\[ \omega = \omega_m = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} < \omega_0 = \sqrt{\frac{k}{m}}. \]

28. As indicated by the cart-with-flywheel example discussed in this section, an unbalanced rotating machine part typically results in a force having amplitude proportional to the square of the frequency \( \omega \). (a) Show that the amplitude of the steady periodic solution of the differential equation

\[ mx'' + cx' + kx = m A_0 \omega^2 \cos \omega t \]

(with a forcing term similar to that in Eq. (17)) is given by

\[ C(\omega) = \frac{m A_0 \omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}. \]

(b) Suppose that \( c^2 < 2mk \). Show that the maximum amplitude occurs at the frequency \( \omega_m \) given by

\[ \omega_m = \sqrt{\frac{k}{m} \left( \frac{2mk}{2mk - c^2} \right)} \cdot \]

Thus the resonance frequency in this case is larger (in contrast with the result of Problem 27) than the natural frequency \( \omega_0 = \sqrt{k/m} \). (Suggestion: Maximize the square of \( C \).)
Automobile Vibrations

Problems 29 and 30 deal further with the car of Example 5. Its upward displacement function satisfies the equation \( mx'' + cx' + kx = cy' + ky \) when the shock absorber is connected (so that \( c > 0 \)). With \( y = a \sin \omega t \) for the road surface, this differential equation becomes

\[
mx'' + cx' + kx = E_0 \cos \omega t + F_0 \sin \omega t
\]

where \( E_0 = ca \) and \( F_0 = ka \).

29. Apply the result of Problem 26 to show that the amplitude \( C \) of the resulting steady periodic oscillation for the car is given by

\[
C = \frac{a \sqrt{k^2 + (c \omega)^2}}{(k - m \omega^2)^2 + (c \omega)^2}.
\]

Because \( \omega = 2\pi v/L \) when the car is moving with velocity \( v \), this gives \( C \) as a function of \( v \).

30. Figure 2.6.12 shows the graph of the amplitude function \( C(\omega) \) using the numerical data given in Example 5 (including \( c = 3000 \) N·s/m). It indicates that, as the car accelerates gradually from rest, it initially oscillates with amplitude slightly over 5 cm. Maximum resonance vibrations with amplitude about 14 cm occur around 32 mi/h, but then subside to more tolerable levels at high speeds. Verify these graphically based conclusions by analyzing the function \( C(\omega) \). In particular, find the practical resonance frequency and the corresponding amplitude.

![Figure 2.6.12. Amplitude of vibrations of the car on a washboard surface.](image)

2.7 Electrical Circuits

Here we examine the \( RLC \) circuit that is a basic building block in more complicated electrical circuits and networks. As shown in Fig. 2.7.1, it consists of

- A **resistor** with a resistance of \( R \) ohms,
- An **inductor** with an inductance of \( L \) henries, and
- A **capacitor** with a capacitance of \( C \) farads

in series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of \( E(t) \) volts at time \( t \). If the switch shown in the circuit of Fig. 2.7.1 is closed, this results in a current of \( I(t) \) amperes in the circuit and a charge of \( Q(t) \) coulombs on the capacitor at time \( t \). The relation between the functions \( Q \) and \( I \) is

\[
\frac{dQ}{dt} = I(t).
\]

We will always use mks electric units, in which time is measured in seconds.

According to elementary principles of electricity, the **voltage drops** across the three circuit elements are those shown in the table in Fig. 2.7.2. We can analyze the behavior of the series circuit of Fig. 2.7.1 with the aid of this table and one of Kirchhoff’s laws:

The (algebraic) sum of the voltage drops across the elements in a simple loop of an electrical circuit is equal to the applied voltage.

As a consequence, the current and charge in the simple \( RLC \) circuit of Fig. 2.7.1 satisfy the basic circuit equation

\[
L \frac{dI}{dt} + RI + \frac{1}{C} Q = E(t).
\]