By definition we say the tangent line to the curve \( f(x) \) at the point \( P(a, f(a)) \) is the line through \( P \) with slope
\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]
* Remember the idea is to get as close to the point \((a, f(a))\), which is why we take the limit as \( h \) approaches to \( a \).

---

Find an equation of the tangent line to the parabola \( y = x^2 - 8x + 9 \) at the point \((3, -6)\)

1. Find difference of quotient

\[
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 8(x+h) + 9 - (x^2 - 8x + 9)}{h}
\]

\[
= \frac{x^2 + 2xh + h^2 - 8x - 8h + 9 - x^2 + 8x - 9}{h}
\]

\[
= \frac{2xh + h^2 - 8h}{h}
\]

\[
= \frac{h(2x + h - 8)}{h}
\]

\[
= 2x + h - 8
\]
(2) Now take the limit as $h$ approaches zero
\[ \lim_{h \to 0} 2x + h - 8 = 2x - 8 \quad \text{Slope of the curve} \]

(3) Find the slope at the specific point $(3, -6)$
\[ 2(3) - 8 = 6 - 8 = -2 \]

(4) Find the equation of the tangent line using algebra
\[ y - y_1 = m(x - x_1) \]
\[ y - (-6) = -2(x - 3) \]
\[ y + 6 = -2x + 6 \]
\[ y = -2x \]

Problems: Find an equation of the tangent line to the curve at the given point.

1. $y = \frac{2x+1}{x+2}$ (1, 1)  
2. $y = \frac{x+3}{x+1}$ (0, 3)
Sketching a specific curve

\[ f(x) = \frac{1}{x} \]

- **Domain**: \((\infty, 0) \cup (0, \infty)\)
- **Range**: \((\infty, 0) \cup (0, \infty)\)

→ Find **x** and **y** intercepts
  - no **x**-int \((0 = \frac{1}{x} \rightarrow a \neq 1)\) This means the function never touches the axis
  - no **y**-int \((y = \frac{1}{0} \text{ undefined})\)
→ Vertical Asymptote (What makes the denominator zero)
  \[ x = 0 \]

→ Horizontal Asymptote (Follow 3 rules)

Given the rational function \( \frac{a x^n + bx^{n-1} + \ldots}{c x^m + dx^{m-1} + \ldots} \)

1. If \( n < m \)
   Horizontal Asymptote at \( y = 0 \)

2. If \( n = m \)
   Horizontal Asymptote at \( y = \frac{a}{c} \)

3. If \( n > m \)
   Horizontal Asymptote up to College algebra we say it doesn't exist
   In calculus we can find it using limits (if possible)

So our horizontal asymptote for \( \frac{1}{x} \) is \( y = 0 \) \((\frac{1}{x} = \frac{1x^0}{x^1})\)

Another nice trick to sketch a curve is knowing their end behavior.
This means finding where the function is increasing and/or decreasing
Use vertical asymptotes as a "cut off point" (Choose a point before and after the VA and see what is happening to the curve)
Putting everything together

\[ f(x) = \frac{1}{x} \]

\[
\begin{array}{c|c}
 x & y \\
-1 & -1 \\
1 & 1 \\
2 & 2 \\
\end{array}
\]

* Checking behavior
  - To the left of vertical asymptote: Choose a couple of points
    \[
    \begin{array}{c|c}
    x & y \\
    -2 & -2 \\
    -1 & -1 \\
    \end{array}
    \]
  - To the right of vertical asymptote: Choose a couple of points
    \[
    \begin{array}{c|c}
    x & y \\
    \frac{1}{2} & \frac{2}{1} \\
    2 & \frac{1}{2} \\
    \end{array}
    \]

\[ (x-1)^2 \] Sketch \( f(x) = \frac{x}{x^2 - x + 1} \)

- Domain: \((-\infty, \infty)\)
- Range: \([-\frac{1}{3}, 1]\)
- \(x\) and \(y\) intercepts
  \[
  \begin{array}{c|c}
  x \text{-int} & y \text{-int} \\
  x = 0 & y = 0 \\
  \end{array}
  \]
- Vertical Asymptote
  None
- Horizontal Asymptote
  \( y = 0 \)

* Finding the range
  \[
  y = \frac{x^2}{x^2 - x + 1} 
  \]
  * Find inverse
    \[
    x = \frac{y}{y^2 + 1} 
    \]
    \[
    x^2 - x y + x - y = 0 
    \]
    \[
    x y^2 - x y - y + x = 0 
    \]
    \[
    x y^2 - (x + 1) y + x = 0 \Leftrightarrow \text{Treat it as a Quadratic} 
    \]
    \[
    y = \frac{(x-1) \pm \sqrt{(-x-1)^2 - 4x(x)}}{2x} 
    \]
    \[
    y = \frac{x+1 \pm \sqrt{x^2 + 2x + 1 - 4x^2}}{2x} 
    \]
    \[
    y = \frac{x+1 \pm \sqrt{-3x^2 + 2x + 1}}{2x} 
    \]
- Find restrictions
  \[
  2x \neq 0 \quad -3x^2 + 2x + 1 \geq 0 
  \]
  \[
  x \neq 0 \quad (-3x - 1)(x - 1) \geq 0 
  \]

* Behavior
  Use zero as "cut-off point"
  \( x = -\frac{1}{3} \quad x = 1 \)
  
  Left side below horizontal
  Right side above horizontal
  \( \text{Asymptote} \)
Problems: Sketch the following:

1. \( f(x) = \frac{x - 1}{x} \)

   Hint: (Horizontal Asymptote
   Doesn't exist)

2. \( f(x) = \frac{2x^2}{x^2 + x - 2} \)

Absolute Value Function: It is defined as a distance function

Sketch \( f(x) = \frac{|x-2|}{x-2} \)

Remember: The absolute value of any number is always positive.

Meaning \( |x| = 3 \) for example has two different answers

\(-3 \) and positive \( 3 \).

In these cases \( \frac{|x-2|}{x-2} \) means: there is a positive and a negative,
therefore we can write absolute value functions as a piecewise function

\[
F(x) = \begin{cases} 
\frac{x-2}{x-2} & x > 0 \\
\frac{-x-2}{x-2} & x < 0 
\end{cases}
\]

Simplify \( F(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x < 0 
\end{cases} \)

Notice: For the original function \( \frac{|x-2|}{x-2} \)

\( F(x) \) is undefined at \( x = 2 \).

Since this part is canceled out in the process we must exclude this point with an open circle,
as \( x = 2 \) is not part of our domain to begin with.

Problem:

Sketch \( f(x) = \frac{3x - |x|}{x} \)
Limits Roughly speaking we can say a limit is a value of \( f(x) \) as \( x \) approaches a. This value can or cannot be included.

Notice that the \( \lim_{x \to 1} f(x) = 1 \) even though \( x \neq 1 \).

For a limit to exist, it must be approach from both sides.

From Trigonometry recall the \( \sin x \) function.

1. As \( x \) approaches \( \frac{\pi}{2} \), \( f(x) \) approaches 1 (or \( \lim_{x \to \frac{\pi}{2}} \sin x = 1 \)).

2. Find \( \lim_{x \to \pi} \sin x = \)

\[ \lim_{x \to \pi} \tan x = \text{DNE} \]

Notice as \( x \to \frac{\pi}{2} \) from the positive side, \( f(x) \) is approaching \( -\infty \) and as \( x \to \frac{\pi}{2} \) from the negative side, \( f(x) \) is approaching \( \infty \).

Since \( f(x) \) is not approaching the same values from both sides of \( \frac{\pi}{2} \), we say the limit does not exist (DNE).
1) Find \( \lim_{x \to \infty} \arctan(x) = \frac{\pi}{2} \)

2) Find a) \( \lim_{x \to 2} \arcsin(x) = \)  

b) \( \lim_{x \to 0} \arctan(x) = \) 

* Even though \( \frac{\pi}{2} \) is not in the range of \( \arctan(x) \), the limit at \( \frac{\pi}{2} \) exists. As \( x \) approaches infinity (\( x \) gets bigger), the function gets close to \( \frac{\pi}{2} \) but never touches it.

**Parametric Equations**: Are equations where \( x \) and \( y \) are given in terms of a third variable, \( t \). Each value of \( t \) determines a point \((x,y)\) and by tracing each point we create a curve called a **parametric curve**. (Parametric curves have direction)

Graph and eliminate parameters to find a cartesian equation
\[
x = t^2 - 2t \\
y = t + 1
\]

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<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
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<tbody>
<tr>
<td>-2</td>
<td>8</td>
<td>-1</td>
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<tr>
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<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

Direction is determined by \( t \). How is \( t \) moving in this case, in the positive direction?
To eliminate the parameter,
\[ x = t^2 - 2t \quad y = t + 1 \]

1. Solve for \( t \) in one of the equation (Avoid powers greater than 1)
   \[ x = t^2 - 2t \quad y = t + 1 \]
   \[ \downarrow \]
   \[ t = 1 - y \]

2. Plug into the other equation
   \[ x = (1-y)^2 - 2(1-y) \]
   \[ x = 1 - 2y + y^2 - 2 + y \]
   \[ x = -3 - y + y^2 \]
   \[ x = y^2 - y - 3 \]

   Notice how this is in fact a parabola just like the graph from the first part.

**Problems**: Sketch and eliminate the parameter to find the Cartesian equations of the following:

1. \( x = 3 - 4t \), \( y = 2 - 3t \)

2. \( x = e^t - 1 \), \( y = e^{2t} \)

3. \( x = \sin t \), \( y = \cos t \) (Hint: Use trig properties)
   \[ 0 < t < \frac{\pi}{2} \]